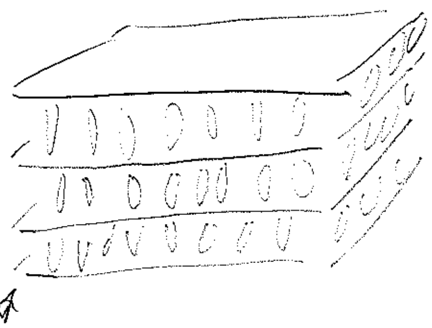
Lecture 7 Elasticity of smectics (Sm A)



$\updownarrow d$

Density $\rho(z) = \rho_0 + \sum_n \left(\Psi_n e^{i n q_0 z} + c.c\right)$

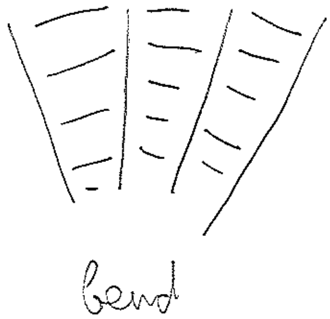
where $q_0 = \frac{2\pi}{d}$

Usually only the first harmonic is important. Then the order parameter Ψ_1 is a complex number $\Psi_1 = |\Psi_1| e^{-i q_0 u}$

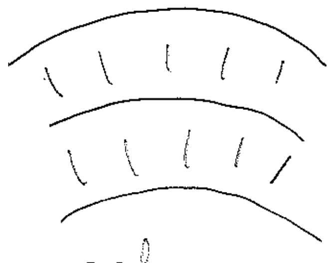
The planes can be interpreted as the planes of constant phase of the density wave.

$$\phi = q_0 z - q_0 u = 2\pi m$$

In the smectic A director $\vec{n}$ is normal to the layers. As a result twist and bend are more costly then splay because they cannot be produced by the constant layer spacing



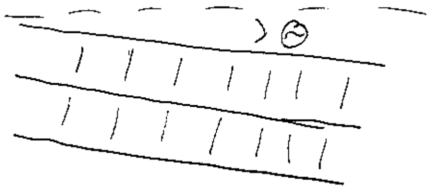
bend



splay

Elastic energy should contain $\left(\frac{\partial u}{\partial z}\right)^2$ term which is equivalent to compression.

Since uniform rotation around the axis in the xy plane doesn't change energy but produces $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$



$u = \theta x$

energy should not contain $(\nabla_\perp u)^2$. Then the leading term with $\nabla_\perp$ and u is $(\nabla_\perp^2 u)$ and we obtain

$$F_{el} = \frac{1}{2}\left[B\left(\frac{\partial u}{\partial z}\right)^2 + K_1 \left(\nabla_\perp^2 u\right)^2\right] \quad (1)$$

Peierls (1935)
Landau (1937)

Note that K_1 is the splay elastic constant of the Frank energy. Since $\nabla_\perp u = -\delta \vec{n} \Rightarrow$

$$(\nabla_\perp^2 u)^2 = (\operatorname{div} \vec{n})^2$$

One can rewrite the energy that treats the displacement u and the Frank director simultaneously. If the layers and the molecules are rotated rigidly there should be no change in energy. However, there will be energy cost if the molecules are rotated away from the normal to the layers. Thus there should be a term in the energy $\propto (\nabla_\perp u + \delta\vec{n})^2$

$$F_{el} = \frac{1}{2}\left[B\left(\frac{\partial u}{\partial z}\right)^2 + D\left(\nabla_\perp u + \delta n\right)^2 + \right.$$
$$\left. + K_1(\vec{\nabla}\cdot\vec{n})^2 + K_2(\vec{n}\cdot \mathrm{rot}\,\vec{n})^2 + K_3[n\times \mathrm{rot}\, n]^2 \right. \qquad (2)$$

Minimizing with respect to δn we obtain $\delta n = -\nabla_\perp u$, terms with K_2, K_3 contain higher order derivatives like $\left(\nabla_\perp u \ \frac{\partial^2 u}{\partial z \partial r_\perp}\right)^2$ and $\left(\frac{\partial^2 u}{\partial r_\perp \partial z}\right)^2$. Ignoring them Eq (2) transforms into Eq (1).

$\delta n = -\nabla_\perp u$ corresponds to splay deformation. The director in a twist or bend deformation is perpendicular to $\nabla_\perp$ => For such deformations a term $D(\delta n)^2$ appears. As a result twist and bend are "gapped" and expelled with the length $\lambda_2 = \sqrt{\frac{K_2}{D}}$ (twist and $\lambda_3 = \sqrt{\frac{K_3}{d}}$ (bend)

In the same way for columnar phases displacement $\vec{u}$ is two dimensional and elastic energy is

$$F_{el} = \frac{\mu}{4}\left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha\right)^2 + \frac{\lambda}{2}(\mathrm{div}\, u)^2 + \frac{B}{2}\left(\frac{\partial^2 u}{\partial z^2}\right)^2$$

$\frac{\partial u}{\partial z}$ corresponds to rotation thus $\left(\frac{\partial u}{\partial z}\right)^2$ is absent

Fluctuations in liquid crystals

Nematics:

$$F \propto q^2 \delta n^2 \Rightarrow$$

$$|\delta n_q|^2 \sim \frac{T}{(\overleftrightarrow{K}\, q)^2}$$

In 3d

$$\langle(\delta n(r) - \delta n(0))^2\rangle \sim T \int \frac{d^3q\,(1-\cos qr)}{\overleftrightarrow{K}\, q^2} = \mathrm{const}$$

$\Rightarrow$ True long range order

For <u>smectics</u>

$$F = \frac{1}{2}\left(B q_z^2 u^2 + K_1 q_\perp^4 u^2\right) \Rightarrow$$

$$u_q^2 = \frac{T}{B q_z^2 + K_1 q_\perp^4} \Rightarrow$$

$$\langle(\delta u(r))^2\rangle = T \int \frac{1-\cos(\vec{q}\cdot\vec{r})}{B q_z^2 + K_1 q_\perp^4}\, dq_z\, d^2q_\perp \sim \frac{T}{\sqrt{BK_1}} \ln\left(\frac{z}{z_0} + \frac{r^2}{r_0^2}\right)$$

Thus – quasi-long-range order (Peierls, Landau)

Power law behaviour of the Bragg peak

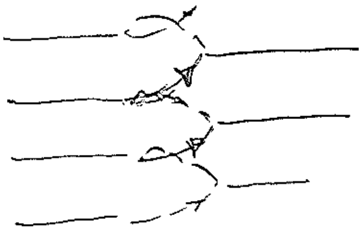
For columnar phases

$$\langle\delta u(\vec{r})^2\rangle = T \int \frac{1-\cos\vec{q}\vec{r}}{B q_z^4 + C q_\perp^2}\, dq_z\, d^2q_\perp = \text{const}$$

$\Rightarrow$ <u>True long range order</u>

Dislocations in smectics

Screw $\vec{b} \parallel \vec{\tau}$



edge $\vec{b} \perp \vec{\tau}$

1. Screw dislocation   Pershan (1974)

$, \delta\vec{n} \perp \vec{n}_0$

$$F_{el} = \frac{D}{2}\left(\vec{\nabla} u + \delta n\right)^2 + \frac{K_1}{2}\left(\vec{\nabla}\cdot\delta n\right)^2 + \frac{K_2}{2}\left[\nabla\times\delta n\right]^2$$

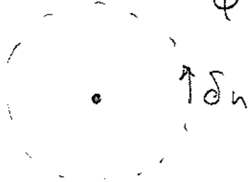
minimizing with respect to u and δn we obtain

$$\nabla\cdot\left(\nabla u + \delta n\right) = 0$$

$$-K_1 \nabla\left(\vec{\nabla}\cdot\delta\vec{n}\right) + K_2 \nabla\times\left[\nabla\times\delta n\right] + D\left(\nabla u + \delta n\right) = 0$$

Since $\oint \nabla u\cdot dl = -b = -d$ we can choose $u = \frac{d\varphi}{2\pi} \Rightarrow$

$\nabla^2 u = 0 \Rightarrow \nabla\cdot\delta n = 0$ thus $\delta n \parallel e_\varphi$



We can define $\vec{Q} = \delta \vec{n} + \vec{\nabla} u = Q(\rho)\, \vec{e}_\varphi$

$$K_2 \operatorname{rot} \operatorname{rot} Q + D Q = 0$$

Because $\operatorname{div} Q = 0$ we can rewrite as

$$\nabla^2 \vec{Q} - \lambda_2^{-2} \vec{Q} = 0$$

with $\lambda_2 = \sqrt{\frac{K_2}{D}}$ – twist length or

$$\rho^2 Q'' + \rho Q' - \left[\left(\frac{\rho}{\lambda_2}\right)^2 + 1\right] Q = 0$$

For $\rho \to 0$ δn is regular but ∇u singular $\Rightarrow$ $Q = \frac{d}{2\pi\rho}$. Full solution

$$Q = \frac{d}{2\pi\lambda_2} K_1\left(\frac{\rho}{\lambda_2}\right)$$ (like vortex in superconductors)

For $\rho \to \infty$, $Q(\rho) \sim \frac{d}{2\sqrt{\pi\lambda_2\rho}} e^{-\rho/\lambda_2} \to 0$

$\delta n = -\nabla u$, δn screens ∇u

distorsions are localized within λ_2

Energy is finite and $\propto \ln \frac{\lambda_2}{r_0}$

Edge dislocation (de Gennes 72)

Only splay $F = \frac{B}{2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{K_1}{2}\left(\nabla_\perp^2 u\right)^2$

minimizing

$$-B\frac{\partial^2 u}{\partial z^2} + K_1 \nabla_\perp^2(\nabla_\perp^2 u) = 0$$

Let us define $\vec{m} = \vec{\nabla} u \Rightarrow$ (3)

$$-\frac{\partial m_z}{\partial z} + \lambda^2 \nabla_\perp^2 (\nabla_\perp \cdot m_\perp) = 0 \, , \quad \lambda = \sqrt{\frac{K_1}{B}}$$

since $\oint \vec{\nabla} u \, dl = -b = \int \mathrm{rot}\, m \, dS \Rightarrow$

$$\mathrm{rot}\, \vec{m} = -b\, \delta^2(\rho) \qquad (4)$$

dislocation along y, then only x, z $\Rightarrow$

$$\left.\begin{array}{ll} \text{Eq.}(4) \Leftrightarrow & i(q_z m_x - q_x m_z) = -b \\ \text{Eq.}(3) \Leftrightarrow & q_z m_z + \lambda q_x^3 m_x = 0 \end{array}\right\} \Rightarrow$$

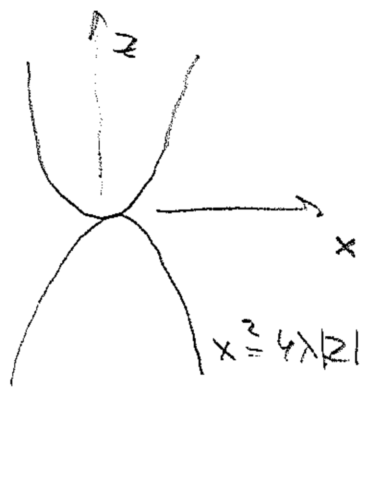
$$\Rightarrow \quad m_x = -\frac{i b\, q_z}{q_z^2 + \lambda^2 q_x^4}$$

$$m_x(x,z) = -b \int \frac{i q_z}{q_z^2 + \lambda^2 q_x^4} e^{i(q_x x + q_z z)} \frac{dq_z\, dq_x}{(2\pi)^2} =$$

$$= \mathrm{sign}(z) \cdot \frac{b}{4\pi} \int dq_x\, e^{-\lambda q_x^2 |z| + i q_x x} \Rightarrow$$

$$\frac{\partial u}{\partial x} = m_x = \pm \frac{b}{4\sqrt{\pi \lambda |z|}} e^{-\frac{x^2}{4\lambda|z|}}$$

$$\frac{\partial u}{\partial z} = \pm \lambda \frac{\partial^2 u}{\partial x^2} = -\frac{b\, x}{8(\pi\lambda)^{1/2} |z|^{3/2}} e^{-\frac{x^2}{4\lambda|z|}}$$



all deformation inside the parabola

finite energy