BCS

$$H = \sum_{k,\sigma} c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{k,k'} b^{\dagger}_{k} b_{k'}$$

$$b^{\dagger}_{k} = c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow}$$

If $b^{\dagger} \rightarrow$ Cooper pair creation op.

then, in gs. for $V_{kk'} < 0$ $\rightarrow$ gs. will have no pair $(k\uparrow, -k\downarrow)$ occupied by a single $\bar{e}$.

Pair states are either empty or doubly occupied!

In this case, $\tilde{H} \rightarrow \sum_{k} 2\varepsilon_{k} b^{\dagger}_{k} b_{k} + \sum_{kk'} V_{kk'} b^{\dagger}_{k} b_{k'}$

$\rightarrow$ looks like a bosonic hamiltonian!

However, b + $b^{\dagger}$ are not bosonic.

$$[b_{k}, b_{k'}] = [b^{\dagger}_{k}, b^{\dagger}_{k'}] = 0.$$

$$[b_{k}, b^{\dagger}_{k'}] = [1 - \underbrace{c^{\dagger}_{k\uparrow} c_{k\uparrow} - c^{\dagger}_{-k\downarrow} c_{-k\downarrow}}_{\text{Pauli blocking!}}] \delta_{kk'}.$$

$$+ \left(b^{\dagger}_{k}\right)^{2} = 0 \rightarrow \text{so not bosonic.}$$

(not easy to diagonalize etc.)

Mean field hypothesis!

$$b_{k} = \langle b_{k} \rangle + \underbrace{(b_{k} - \langle b_{k} \rangle)}_{\delta b_{k}}.$$

Neglecting terms proportional to $\delta b_{k} \delta b^{\dagger}_{k}$

$$H_{MF} = \sum_{k\sigma} \varepsilon_{k} c^{\dagger}_{k\sigma} c_{k\sigma} + \sum_{kk'} V_{kk'} \left[\langle b_{k'} \rangle b^{\dagger}_{k} + \langle b^{\dagger}_{k} \rangle b_{k'} - \langle b^{\dagger}_{k} \rangle \langle b_{k'} \rangle \right]$$

Rewrite as

$$H^{MF} = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_k \left[\Delta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \Delta^*_k c_{-k\downarrow} c_{k\uparrow} \right)$$

$$- \sum_{kk'} V_{kk'} \langle b^\dagger_k \rangle \langle b_k \rangle$$

where
$$\boxed{\Delta_k = \sum_k V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle}$$

- Notice that $[H^{MF}, N] \neq 0$ number not conserved!

$N = \sum_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma}$ $\Longrightarrow$ work in grand canonical ensemble $\boxed{K = H^{MF} - \mu N}$

Solution to MF equations:

$$K = \sum_k \left(c^\dagger_{k\uparrow} \; c_{-k\downarrow} \right) \begin{bmatrix} \xi_k & \Delta_k \\ \Delta^*_k & -\xi_k \end{bmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + K_0$$

$\xi_k = \varepsilon_k - \mu$

$$K_0 = \underbrace{\sum_k \xi_k - \sum_{kk'} V_{kk'} \langle c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \rangle \langle h.c. \rangle_{k'}}_{\text{constant.}}$$

Diagonalize via unitary transform

$$\begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} = U_k \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix} \qquad U_k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k e^{i\phi_k} \\ \sin\theta_k e^{-i\phi_k} & \cos\theta_k \end{bmatrix}$$

with $\{ \gamma_{k\sigma}, \gamma^{\dagger}_{k'\sigma'} \} = \delta_{kk'} \delta_{\sigma\sigma'}$

This transformation mixes particle & hole states.

for each 'k' mode

$\tilde{K} = U^{\dagger} K U$ & we want $\tilde{K}$ to be diagonal.

$\Longrightarrow \phi = \arg(\Delta)$ and $\tilde{K} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$

$$2\xi \sin\theta \cos\theta = |\Delta|(\cos^2\theta - \sin^2\theta).$$

Bogoliubov-Valatin transf.

$$\tan 2\theta = \frac{|\Delta|}{\xi} \qquad \cos 2\theta = \frac{\xi}{E} \qquad \sin 2\theta = \frac{|\Delta|}{E}$$

$$E = \sqrt{\xi^2 + |\Delta|^2}$$

Restoring "k"

$$\phi_k = \arg(\Delta_k) \qquad \tan 2\theta_k = \frac{|\Delta_k|}{\xi_k}.$$

$$\cos 2\theta_k = \frac{\xi_k}{E_k} \qquad \sin 2\theta_k = \frac{|\Delta_k|}{E_k} \qquad E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$$

If Δ_k has only a weak dependence on k,

$E_k \rightarrow$ dispersion of excitations has minimum at $\xi_k = 0$ i.e. $k = k_F$

$|\Delta_k| \rightarrow$ superconducting gap!.

$$\hat{K} = \sum_{k\sigma} E_k \gamma^{\dagger}_{k\sigma} \gamma_{k\sigma} + \sum_k (\xi_k - E_k) - \sum_{kk'} V_{kk'} \langle\,\rangle\langle\,\rangle$$

What is the ground state of such a system?

$\Rightarrow$ GS $\Rightarrow$ $\gamma_{k\sigma}|G\rangle = 0$

$$\Rightarrow \quad |G\rangle = \prod_k \Big[\underbrace{\cos\theta_k}_{u_k} - \underbrace{\sin\theta_k\, e^{i\phi_k}}_{v_k} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\Big]|0\rangle .$$

Normal metallic GS

$\langle G|G\rangle = 1$.

$$\rightarrow |Q\rangle = \prod_{k\sigma}^{\varepsilon_k<\mu} c^\dagger_{k\sigma}|0\rangle = \prod_k^{\varepsilon_k<\mu} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}|0\rangle$$

same form as $S \subset |G\rangle$

with $u_k = 0$ $\varepsilon_k<\mu$

$u_k = 1$ $\varepsilon_k>\mu$

$\parallel^{ly}$ for v_k.

$$\gamma_{k\sigma} = \cos\theta_k\, c_{k\sigma} + \sigma \sin\theta_k e^{i\phi_k} c^\dagger_{-k-\sigma}$$

~~When~~ If $\boxed{\Delta_k = 0}$ + $\xi_k < 0$ for $k<k_F$

Choose $\phi_k = 0$ $\quad\xi_k > 0$ for $k>k_F$

$$\cos\theta_k = \theta(k-k_F) \qquad \sin\theta_k = \theta(k_F-k)$$

then $|G\rangle \longrightarrow$ describes filled Fermi sphere with upto $k = k_F$.

$k<k_f$ $\quad \gamma^\dagger_{k\sigma} = \sigma\, c_{-k-\sigma}$

$k>k_f$ $\quad \gamma^\dagger_{k\sigma} = c^\dagger_{k\sigma}$

elementary excns. are holes below k_F + electrons above k_F!

Self-consistency

$$N = \sum_{k\sigma} \langle c^\dagger_{k\sigma} c_{k\sigma} \rangle$$

$$\Delta_k = \sum_{k'} V_{kk'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle .$$

We thus have

$$\langle c^\dagger_{k\sigma} c_{k\sigma} \rangle = \cos^2\theta_k f_k + \sin^2\theta_k (1-f_k)$$

$$= \frac{1}{2} - \frac{\xi_k}{2E_k} \tanh\left(\beta \frac{E_k}{2}\right)$$

$$f_k = \langle \gamma^\dagger_{k\sigma} \gamma_{k\sigma} \rangle = \frac{1}{e^{\beta E_k}+1} \longrightarrow \text{fermi fn. at temp } T = \frac{1}{\beta}.$$

$$\langle c_{-k-\sigma} c_{k\sigma} \rangle = \sigma \sin\theta_k \cos\theta_k e^{i\phi_k} [2f_k - 1] \qquad \boxed{\sigma = \pm 1}$$

$$= \frac{-\sigma \Delta_k}{2E_k} \tanh\left(\beta \frac{E_k}{2}\right)$$

$\boxed{\text{At } T=0}$

$$N = \sum_k \left[1 - \frac{\xi_k}{E_k}\right] \qquad [\text{th} \longrightarrow 1]$$

$$\Delta_k = -\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}} \longrightarrow \boxed{\text{BCS Gap Eq.}}$$

X^n model

$$: \quad V_{kk'} = \begin{cases} -v/V? & |\xi_k| < \hbar\omega_D \\ & |\xi_{k'}| < \hbar\omega_D \\ 0 & \end{cases}$$

[like in Cooper pair problem]

$v > 0$ of X^n attractive in an energy band $\hbar\omega_D$.

$\omega_D \to$ Debye freq. for phonons

$$\Delta_k = \begin{cases} \Delta e^{i\phi} & |\xi_k| < \hbar\omega_D . \\ 0 & \text{otherwise} . \end{cases}$$

$\Delta \to$ real then.

$$\Delta = v \int \frac{d^3k}{(2\pi)^3} \frac{\Delta}{2E_k} \theta(\hbar\omega_D - |\xi_k|).$$

assume
$g(\xi)$ varies slowly around $\mu \lessapprox \varepsilon_F$.
($\mu \simeq \varepsilon_F$ true at low T).

$$= \frac{1}{2} v\, g(\varepsilon_F) \int_0^{\hbar\omega_D} d\xi \frac{\Delta}{\sqrt{\xi^2 + \Delta^2}}$$

Trivial soln: $\Delta = 0$!

Non-trivial soln:

$$1 = \frac{v g(\varepsilon_F)}{2} \int_0^{\frac{\hbar\omega_D}{\Delta}} \frac{ds}{\sqrt{1+s^2}} = \frac{1}{2} v g \sinh^{-1}\left(\frac{\hbar\omega_D}{\Delta}\right).$$

Zero temp gap

$$\Delta_0 \simeq 2\hbar\omega_D e^{-\frac{2}{g(\varepsilon_F) v}}.$$

[difference in argument of exponent as compared with the Cooper problem]. $\to e^{-\frac{4}{g(\varepsilon_F) v}}$. !

$g(\varepsilon_F) = 2N(0) \to$ for each spin
$\downarrow$
DoS for both spins

$\boxed{\text{Justifies simple Cooper solution.}}$

First application

$$\frac{E^S}{\cancel{V}} = \langle G | K^{MF}_{BCS} | G \rangle = \sum_k \left[\xi_k - E_k + \frac{|\Delta_k|^2}{2E_k} \right]$$

Subtract energy of metallic phase i.e. $\Delta_k = 0$.

$$E^M = \sum_{k < k_F} \xi_k \, \theta(k_F - k). \quad \text{[just the kinetic term]}$$

$$\frac{E^S - E^M}{V} = 2 \sum_k \left[(\xi_k - E_k) \, \theta(\xi_k) \, \theta(\hbar\omega_D - \xi_k) \right.$$
$$\left. + \sum_k \frac{\Delta_0^2}{2E_k} \, \theta(\hbar\omega_D - |\xi_k|) \right].$$

(using nature of xn pot.)

$$= g(\varepsilon_F) \, \Delta_0^2 \int \ldots$$

$$\approx -\frac{1}{4} g(\varepsilon_F) \Delta_0^2 \equiv -\frac{B_c^2(0)}{2\mu_0}.$$

$$\frac{B_c^2}{2\mu_0} = \frac{H_c^2}{8\pi}$$

$$\Rightarrow \boxed{B_c = \frac{\mu_0 g(\varepsilon_F)}{2} \Delta_0.}$$

$$B_c^2 = \frac{\mu_0}{4\pi} H_c^2.$$

$$B_c = \sqrt{\mu_0}$$

$$= \cancel{V} g(\varepsilon_F) \Delta_0^2 \int_0^{\hbar\omega_D/\Delta_0} ds \left[s - \sqrt{s^2+1} + \frac{1}{2\sqrt{s^2+1}} \right]$$

$$= V \frac{g}{2} \Delta_0^2 \left(x^2 - \sqrt{1+x^2} \right) \approx -\frac{1}{4} V g(\varepsilon_F) \Delta_0^2$$

$$x = \frac{\hbar\omega_D}{\Delta_0}$$

Number and phase

Consider a state:

$$|G(\alpha)\rangle = \prod_k \left[\cos\theta_k - e^{i\alpha} e^{i\phi_k} \sin\theta_k \, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right] |0\rangle$$

$||\rangle$

$$\hat{N}\,|G(\alpha)\rangle = 2i \frac{\partial}{\partial\alpha} |G(\alpha)\rangle .$$

No: of cooper pairs $\hat{M} = \frac{\hat{N}}{2}$ then $\hat{M} = i\frac{\partial}{\partial\alpha}$.

$\alpha + N$ are conjugates!

Projecting $|G(\alpha)\rangle$ onto states of definite particle number

by defining

$$|M\rangle = \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{-iM\alpha} |G(\alpha)\rangle$$

State $|M\rangle$ has $N = 2M$ particles or N cooper pairs.

$$\frac{\langle G(\alpha)|N^2|G(\alpha)\rangle - \langle G(\alpha)|N|G(\alpha)\rangle^2}{\langle G(\alpha)|N|G(\alpha)\rangle} = 2\frac{\int d^3k \sin^2\theta_k \cos^2\theta_k}{\int d^3k \sin^2\theta}.$$

$\sin\theta_k = \Theta(k_F - k)$
$\cos\theta_k = \Theta(k - k_F)$
$= 0$ in fermi liquid regime

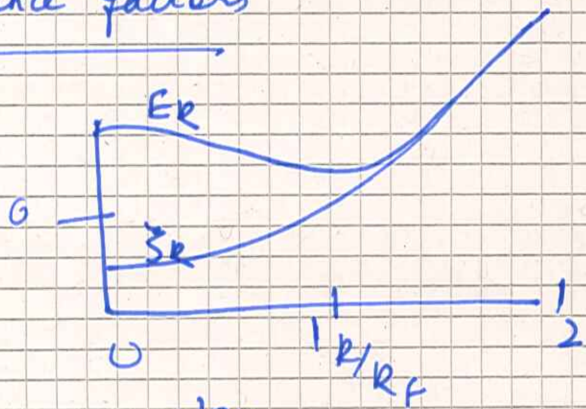
$\therefore (\Delta N)_{RMS} \propto \sqrt{N}$

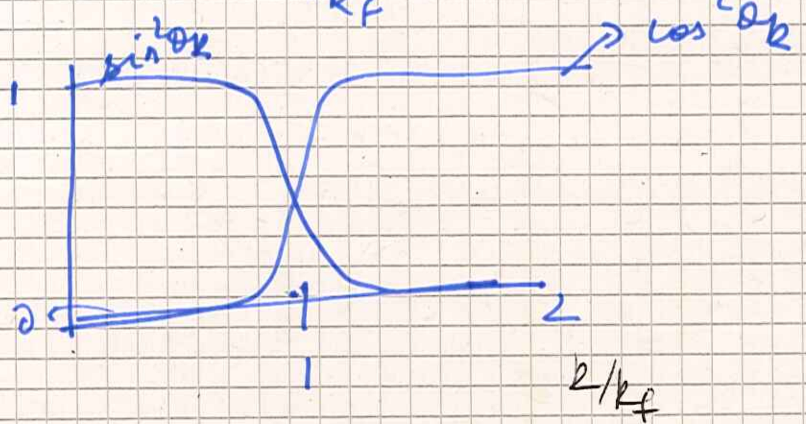
$\Delta\theta \, \Delta N \approx 1$

In Normal phase, θ is indeterminate $\longrightarrow$ $U(1)$ gauge invariance

But in SC phase θ is determined due to $U(1)$ symm breaking!

Coherence factors





$\sin^2\theta_k \to 1$ for $k < k_f$ — width of variation window

$\cos^2\theta_k \to 1$ for $k > k_f$

$$\delta k \approx \frac{\Delta_0}{\hbar v_f} \to \xi^{-1}$$

→ coherence length.

Since $\gamma^\dagger_{k\sigma} = \cos\theta_k c^\dagger_{k\sigma} + \sigma\sin\theta_k e^{-i\phi_k} c_{-k-\sigma}$

$\gamma^\dagger_{k\sigma}$ creates e^- like exc^n in (k,σ) when $\cos\theta_k \to 1$

" hole like exc^n in $(-k,-\sigma)$ when $\sin\theta_k \to 1$

for $|k-k_f| \lesssim \frac{\Delta_0}{\hbar v_f}$, this operator creates a linear combo of e^- + hole states

Typically $\Delta_0 \sim 10^{-4}\varepsilon_f$ — ε_f in metals ~~keV range~~. tens of thousands of Kelvin

$$\delta k \lesssim 10^{-3} k_f$$

Thus the entire physics of s wave SC states takes place within an onion skin at the fermi surface.

Finite temperature solution

Gap eqn: $\Delta_k = -\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}} \tanh \frac{E_{k'}}{2k_B T}$.

clearly $\Delta k \to 0$ as $T \to \infty$. $\sum_{k'} V_{kk'} \Delta_{k'} = -4k_B T \Delta_k$

If V_{kk} is bounded, no solution for $k_B T$ greater than largest eigenvalue of $V_{kk'}$

To find the critical temp, we recast this eqn.

$$1 = \frac{g(\varepsilon_F)\, v}{2} \int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\xi^2+\Delta^2}} \tanh \frac{\sqrt{\xi^2+\Delta^2}}{2k_B T}.$$

T_c is when $\Delta \to 0$.

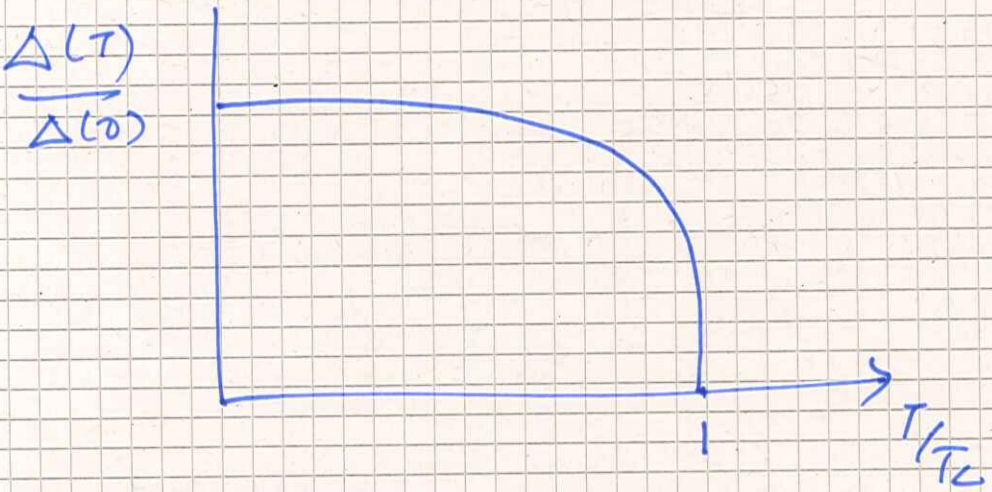
$$\frac{2}{g(\varepsilon_F) V} = \int_0^{\hbar\omega_D} \frac{\tanh \frac{\xi}{2k_B T_c}}{\xi} \, d\xi.$$

$$\simeq \ln \frac{2\gamma \hbar\omega_D}{\pi k_B T_c}$$

$\gamma \to$ Euler const. $(0.577\ldots)$

$$\boxed{k_B T_c \simeq 1.13\, \hbar\omega_D \, e^{-\frac{2}{g(\varepsilon_F) v}}}$$

[Because v is typically very small
BCS theory predicts upper limit for $T_c \sim 30\text{-}40\,K$



$$\boxed{\Delta(0) = 1.76\, k_B T_C}$$

we also know that

$$\Delta_0 \simeq 2\hbar\omega_D \exp - \frac{2}{g(\varepsilon_F) V}$$

$$k_B T_C = \frac{1.13}{2} \Delta(0).$$

$$\boxed{\Delta(0) \simeq 1.76\, k_B T_C}$$

$$2\Delta \simeq 3.5\, T_C.$$ BCS relation

Old s wave SC $\rightarrow 3.0\, T_C < 2\Delta < 4.5\, T_C$. !

(no true for unconventional sc !)

Below T_C $\quad T \rightarrow T_C^-$

$$\Delta(T) \simeq 3.06\, k_B T_C \left[1 - \frac{T}{T_C}\right]^{1/2}.$$

Isotope effect

Things are proportional to $\hbar\omega_D$.

$$\therefore \ln T_c = \ln \omega_D - \frac{2}{g(\varepsilon_F)\,v} + \text{const}.$$

$\omega \sim \sqrt{\frac{k}{m}}$.

If we can vary the mass of ions, via isotopic substitution while not changing $g(\varepsilon_F)$ & v

then $\quad \delta \ln T_c = \delta \ln \omega_D = -\frac{1}{2}\,\delta \ln M.$

$\therefore$ increasing M decreases T_c!

What is the impact of repulsive x^{ns}?

$$V_{kk'} = \begin{cases} v_c - v_p / V & |\xi_k|, |\xi_{k'}| < \hbar\omega_D \\ v_c / V & \text{otherwise}. \end{cases}$$

$$\Delta_k = \begin{cases} \Delta_0 \text{ real} & |\xi_k| < \hbar\omega_D \\ \Delta_1 \text{ - real} & \text{otherwise}. \end{cases}$$

Assume $\underset{\text{coulomb repulsion}}{\overset{\text{phonons}}{v_p > v_c}}$ to have any attraction!

The gap equation leads to 2 eqns. for Δ_0 & Δ_1 now.

$$k_B T_c = 1.134\,\hbar\omega_D \exp -\frac{2}{g(\varepsilon_F)\,v_{eff}}$$

$$v_{eff} = v_p - \frac{v_c}{1 + \frac{1}{2}\,g(\varepsilon_F) \ln B/\hbar\omega_D}\, v_c \qquad B \text{ — bandwidth of electrons}$$

At $T=0$, gap equation,

$$\Delta_0 = \frac{1}{2} g(\varepsilon_F)(V_p - V_c)\int_0^{\hbar\omega_D} d\xi \, \frac{\Delta_0}{\sqrt{\xi^2+\Delta_0^2}} - \frac{1}{2} g(\varepsilon_F) V_c \int_{\hbar\omega_D}^{B} \frac{\Delta_1}{\sqrt{\xi^2+\Delta_1^2}}$$

$$\Delta_1 = -\frac{1}{2} g(\varepsilon_F) V_c \int_0^{\hbar\omega_D} \swarrow \; - \frac{1}{2} g(\varepsilon_F) V_c \int \swarrow$$

$\omega_D \rightarrow$ Debye freq

$B \rightarrow$ Bandwidth

assume $\Delta_{0,1} \ll \hbar\omega_D \ll B$

$$\Delta_1 = -\frac{\frac{1}{2} g V_c \ln\left(\frac{2\hbar\omega_D}{\Delta_0}\right)}{1 + \frac{1}{2} g V_c \ln B/\hbar\omega_D} \Delta_0$$

Soln. only if

$$\frac{2}{g V_p} = \frac{1}{V_p} \ln\frac{2\hbar\omega_D}{\Delta_0}\left[V_p - V_c \cdot \frac{1}{1 + \frac{g}{2}\ln B/\hbar\omega_D V_c}\right]$$

$B \gg \hbar\omega_p$. $\quad e^{\frac{1}{x}} \quad e^{-\frac{1}{x}}$

$e^{-\frac{1}{\cdots}}$ $\quad V_{eff} \downarrow$ & $T_c \downarrow$. $\quad e^{\frac{1}{1/x}}$.

Tc decreases with repulsive x^{ns}.

& so does Δ_0.

$$\lambda = \frac{g}{2} v_p \quad \mu = \frac{g}{2} v_c \quad \mu^* = \frac{\mu}{1+\mu \ln B/\hbar\omega_p}$$

then $\quad k_B T_c = 1.134\, \hbar\omega_p\, e^{-1/\lambda-\mu^*}$

Isotope effect is also modified.