Exercise 1. ***Quantum mutual information***

One way of quantifying correlations between two systems A and B is through their *mutual information* $I(A:B)$.

(a) Consider two qubits A and B in joint state ρ_{AB}.

  (i) Prove that the mutual information of the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is maximal. This is why we say Bell states are *maximally entangled*.

  (ii) Show that $I(A:B) \leq 1$ for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|_A \otimes \sigma_B^1$ (where $0 \leq p \leq 1$).

(b) Consider the so-called *cat state* of four qubits, $A \otimes B \otimes C \otimes D$, that is defined as

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) . \tag{1}$$

Check how the mutual information between A and B changes with the knowledge of the remaining qubits, i.e compute

  (i) $I(A:B)$,

  (ii) $I(A:B|C)$,

  (iii) $I(A:B|CD)$.

(c) Can you give an intuitive explanation for the results in (b)?

Exercise 2. ***Classical and quantum Markov chains***

Three random variables X, Y, Z form a *Markov chain* (also: have the *Markov property*), denoted by $X \leftrightarrow Y \leftrightarrow Z$, if $P_{Z|YX=x} = P_{Z|Y}$ for all $x \in \mathcal{X}$, the alphabet of X. One way of interpreting this is to say that once we know Y, we cannot learn more about Z when learning X.

(a) Show that the Markov property is symmetric, i.e. that it implies $P_{X|YZ=z} = P_{X|Y}$ for all $z \in \mathcal{Z}$.
Remark: This is already suggested by the notation.

(b) Prove that for Markov chains the conditional mutual information is zero, $I(X:Z\,|\,Y) = 0$. This is a mathematical way of stating the interpretation mentioned above.

For quantum states we can define the Markov property as follows. A state ρ_{ABC} on a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ is called a *Markov state* if there exists a CPTP map $\mathcal{T}_{B\to BC}$ from B to BC s.t. $\rho_{ABC} = \mathcal{I}_A \otimes \mathcal{T}_{B\to BC}(\rho_{AB})$.

(c) Explain how this definition can be interpreted in the same way as the classical one.

(d) Prove that the GHZ state, the 3-qubit state $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, is not a Markov state by explicitly showing that a reconstruction map $\mathcal{T}_{B\to BC}$ cannot exist.
Remark: Please do not use (e) here.

(e) Prove that for quantum Markov states $I(A : C \,|\, B) = 0$.
Hint: Use strong subadditivity.