Exercise 1. ***Trace distance and distinguishability***

Suppose you know the density operators of two quantum states $\rho, \sigma \in \mathcal{S}(\mathcal{H})$. Then you are given one of the states at random – it may either be ρ or σ with equal probability. The challenge is to perform a single projective measurement of an observable O on your state and then guess which state that is.

(a) What is your best strategy? In which basis do you think you should perform the measurement? Can you express that measurement using a single projector P?

(b) Show that the probability of guessing correctly can be written as

$$P_{\text{guess}}(\rho \text{ vs. } \sigma) = \frac{1}{2}(1 + \operatorname{tr}[P(\rho - \sigma)]), \tag{1}$$

where P is the appropriate projector from (a).

Just like in the classical case, that can be shown to be equivalent to

$$P_{\text{guess}}(\rho \text{ vs. } \sigma) = \frac{1}{2}[1 + \delta(\rho, \sigma)], \tag{2}$$

where $\delta(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_1$ is the trace distance between the two quantum states and $\|S\|_1 := \operatorname{tr}|S| \equiv \operatorname{tr}[\sqrt{S^\dagger S}]$ the 1-norm for matrices. (You do not have to show this here. Of course you can if you want.)

(c) Given a trace-preserving quantum operation $\mathcal{E}$ (i.e. a CPTP map) and two states ρ and σ, show that

$$\delta(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq \delta(\sigma, \rho). \tag{3}$$

(d) What does (3) imply about the task of distinguishing quantum states?

Exercise 2. ***Fidelity and Uhlmann's Theorem***

Given two states ρ_A and σ_A on $\mathcal{H}_A$ with fixed basis $\{|i\rangle_A\}_i$ and a reference Hilbert space $\mathcal{H}_B$ with fixed basis $\{|i\rangle_B\}_i$, which is a copy of $\mathcal{H}_A$, Uhlmann's theorem claims that the fidelity can be written as

$$F(\rho_A, \sigma_A) = \max_{|\Psi\rangle_{AB}, |\Phi\rangle_{AB}} |\langle\Psi|\Phi\rangle|, \tag{4}$$

where the maximum is over all purifications $|\Psi\rangle_{AB}$ of ρ_A and $|\Phi\rangle_{AB}$ of σ_A on $\mathcal{H}_A \otimes \mathcal{H}_B$. Let us introduce the state $|\psi\rangle_{AB}$ as

$$|\psi\rangle = (\sqrt{\rho} \otimes U_B)\,|\Omega\rangle, \qquad |\Omega\rangle = \sum_i |i\rangle_A \otimes |i\rangle_B, \tag{5}$$

where U_B is any unitary on $\mathcal{H}_B$. We have seen in Exercise Sheet 6 that $|\psi\rangle_{AB}$ is a purification of ρ_A and that any purification of ρ_A can be written in this form.

(a) Use the construction presented in the proof of Uhlmann's theorem to calculate the fidelity between $\sigma'_A = \mathbb{1}_2/2$ and $\rho'_A = p|0\rangle\langle 0|_A + (1-p)|1\rangle\langle 1|_A$ in the 2-dimensional Hilbert space.

Hint: Convince yourself that the vector $|\Omega\rangle$ has the property that $\mathbb{1} \otimes S\,|\Omega\rangle = S^T \otimes \mathbb{1}|\Omega\rangle$ for all linear operators S on $\mathcal{H}_A$.

(b) Give an expression for the fidelity between any pure state and the completely mixed state $\mathbb{1}_n/n$ in the n-dimensional Hilbert space.

Hint: You may want to use a different characterization of the fidelity than the one by Uhlmann for this exercise.

Exercise 3. ***An interpretation of the quantum trace distance***

In Exercise Sheet 3 we have seen an interpretation of the classical trace distance. We have shown that two probability distributions that are ε-close in trace distance allow for a joint distribution s.t. the corresponding reduced random variables differ with probability at most ε. In the quantum case, where probability distributions are replaced by quantum states, say ρ and $\sigma \in \mathcal{S}(\mathcal{H})$, this statement does not have a direct translation. Instead, as we will see in this exercise, there is a similar but different way of interpreting the trace distance.

(a) Thinking of the classical version from Exercise Sheet 3 again, what goes wrong when trying to 'quantize' this interpretation directly? Why does this not work?

Suppose that $\delta(\rho, \sigma) = \varepsilon$. We will show that there is a quantum state $\omega \in \mathcal{S}(\mathcal{H})$ that can be written in two ways,

$$\begin{aligned} \omega &= (1-\varepsilon)\rho + \varepsilon\hat{\rho} \\ &= (1-\varepsilon)\sigma + \varepsilon\hat{\sigma}, \end{aligned} \tag{6}$$

where $\hat{\rho}, \hat{\sigma} \in \mathcal{S}(\mathcal{H})$ are some quantum states.

(b) Use the fact that the operator $\rho - \sigma$ can be decomposed into $\rho - \sigma = R - S$, where both R and S are positive operators with mutually orthogonal support, and $\mathrm{tr}[R] = \varepsilon$, to construct ω.

The above statement has two interpretations: (i) there exists a state ω that behaves as if it was ρ with probability $1-\varepsilon$; (ii) the same state ω behaves exactly like σ with probability $1-\varepsilon$.

(c) Can you construct a classical example of this interpretation in the language of density operators to illustrate the connection to the classical version?