Exercise 1. ***Distinguishing channels***

The setting is the following: With equal probabilities you are given either the identity channel I on some finite alphabet $\mathcal{X}$ or an arbitrary channel W on the same alphabet, without knowing which. In terms of conditional probability distributions the channels are described by

$$I(x \,|\, x') = \delta_{xx'} \quad \text{and} \quad W(x \,|\, x'), \tag{1}$$

for $x, x' \in \mathcal{X}$. You are allowed to use the given channel once, possibly with a stochastic (randomized) input, and then asked which channel was used.

(a) The error probability of a channel W is defined as $P_{\text{error}}(W) := \max_{x\in\mathcal{X}} \big(1 - W(x \,|\, x)\big)$. Argue why this is a sensible definition.

(b) Using properties of the trace distance of probability distributions previously derived in Exercise Sheet 1, Exercise 1(c), show that the probability of guessing the channel correctly in the above scenario is

$$P_{\text{guess}}(I \text{ vs. } W) = \frac{1}{2}\big(1 + P_{\text{error}}(W)\big). \tag{2}$$

Exercise 2. ***Source coding***

In the lecture we have seen a channel coding theorem with a more or less tricky derivation involving interesting ideas which were needed to get to the final result. In this exercise we will consider source coding, the challenge of which is to compress some input described by a RV X to another RV Y. One can think of this as the task to find the optimal channel $P_{Y|X}$ such that all information contained in X can be retrieved from Y, but the alphabet $\mathcal{Y}$ of Y should be as small as possible. Instead of phrasing the problem in a technical way, we are here taking a simpler and more conceptual approach.

Consider a k-bit string described by the RV X distributed P_X. We would like to compress this to a l-bit string described by a RV Y distributed P_Y. The goal is to find the minimal l such that Y contains (almost) all information from X.

(a) Suppose you are not interested in many but just in a single use of the compressed source. Furthermore you will not tolerate any errors. How small can l be chosen in this case?

(b) If you allow for some small error ε, what is the answer now?

(c) Often one is interested in many (independent) uses of the source. Using asymptotic equipartition results for max-entropies, show that for N i.i.d. uses of the source the optimum compression requires $NH(X)_{P_X}$ bits in the limit $N \to \infty$.

This is called Shannon's source coding theorem. Interestingly, using single-shot quantities and results about asymptotic equipartition this result becomes almost trivial.

Exercise 3. ***Getting used to the Bloch representation***

In this exercise we will see that any density operator of a qubit can be written as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}), \tag{3}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices and $\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3, |\vec{r}| \leq 1$ is the so-called Bloch vector that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere. Thereby we go through a few properties of density matrices describing states of a qubit using the Bloch representation. This way of expressing qubit states is very convenient and frequently used in quantum information theory.

Hint: The (anti-)commutation relations for Pauli matrices will be helpful.

(a) Using Eq. (3):

(i) Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases, $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|\circlearrowleft\rangle, |\circlearrowright\rangle\}$.

(ii) Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

(b) Show that the operator ρ defined in Eq. (3) is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:

(i) Hermiticity: $\rho = \rho^\dagger$.

(ii) Positivity: $\rho \geq 0$.

(iii) Normalisation: $\mathrm{tr}(\rho) = 1$.

(c) Now do the converse: show that any qubit density operator may be written in the form of Eq. (3).

(d) Check that the surface of the ball is formed by all the pure states, i.e.

$$\rho \text{ pure } \Leftrightarrow \rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \text{ with } |\vec{r}| = 1. \tag{4}$$