Exercise 1. ***Classical Klein-Gordon field***

The Lagrangian of a classical real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

results in equations of motion whose general solution can be written as

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_{\vec{p}}} \left(a_{\vec{p}}\, e^{-i\,px} + a^*_{\vec{p}} e^{i\,px}\right)$$

a) Starting from the energy-momentum tensor show that $\vec{P} = -\int d^3\vec{x}(\vec{\nabla}\phi)\dot{\phi}$.

b) Express $\vec{P}$ in terms of the coefficients $a_{\vec{p}}$ and $a^*_{\vec{p}}$ and interpret the result.

Exercise 2. ***Representations of*** $J^{\mu\nu}$

The Lie algebra of the Lorentz group is given by

$$[J^{\mu\nu}, J^{\lambda\kappa}] = i(g^{\mu\kappa}\, J^{\nu\lambda} + g^{\nu\lambda}\, J^{\mu\kappa} - g^{\nu\kappa}\, J^{\mu\lambda} - g^{\mu\lambda}\, J^{\nu\kappa}) \tag{1}$$

(a) Show that the generators of the vector representation

$$(J^{\mu\nu}_V)^\rho{}_\sigma \equiv i(g^{\mu\rho}\, g^\nu{}_\sigma - g^{\nu\rho}\, g^\mu{}_\sigma)$$

satisfy the Lie algebra, Eq.(1).

(b) Show that the operators

$$L^{\mu\nu} \equiv i(x^\mu\, \partial^\nu - x^\nu\, \partial^\mu)$$

satisfy the Lie algebra, Eq.(1).

Exercise 3. ***Weyl spinors and the Dirac equation***

(a) Let $\psi_R(0)$ be a right-handed Weyl spinor for a particle of mass m at rest. Using the transformation properties of ψ_R under a Lorentz boost, show that

$$\psi_R(p) = \frac{E + m + \vec{p}\cdot\vec{\sigma}}{\sqrt{2m\,(E+m)}}\, \psi_R(0) \tag{2}$$

where $\psi_R(p)$ is a right-handed Weyl spinor for a particle with momentum $p^\mu = (E, \vec{p})$.

(b) Deduce directly from Eq.(2) that the corresponding relation for a left-handed Weyl spinor is given by

$$\psi_L(p) = \frac{E + m - \vec{p}\cdot\vec{\sigma}}{\sqrt{2m\,(E+m)}}\, \psi_L(0) \tag{3}$$

(c) Combining Eqs.(2) and (3) with $\psi_R(0) = \psi_L(0)$ derive the Dirac equation

$$\begin{pmatrix} -m & E - \vec{p}\cdot\vec{\sigma} \\ E + \vec{p}\cdot\vec{\sigma} & -m \end{pmatrix} \begin{pmatrix} \psi_L(p) \\ \psi_R(p) \end{pmatrix} = 0 \tag{4}$$