5. Interacting field theory

5.1. ϕ^4 theory

Serves as prototype for interacting field theory

$$\mathcal{L} = +\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m_0^2}{2}\phi^2 - \underbrace{\frac{\lambda_0}{4!}\phi^4}$$

$\mathcal{L}_{int}$ interaction $\rightarrow$ $H_{int} = -\int d^3\vec{x}\, \mathcal{L}_{int}$

λ_0 ~ coupling, 4! for convenience → later

Can still define $\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot\phi}$ and impose ETCR but

basically no interacting field theory can be solved exactly

(few academic exceptions, e.g. QED in 1+1 dim)

→ will have to rely on perturbation theory, λ_0 ~ "small"

EoM $\quad \partial_\mu\partial^\mu\phi + m_0^2\phi + \frac{\lambda_0}{3!}\phi^3 = 0 \quad$ → not linear, not simple plane wave solution

→ no particle interpretation

basic assumptions about spectrum:

- theory has a ground state $|\Omega\rangle$ ($= |0\rangle$ in free case) and $\langle\Omega|\phi(x)|\Omega\rangle = 0$
- we have first excited states (~ "single particle" states) with $P^2|p\rangle = m^2|p\rangle$ and $m^2 \geq 0$
- other eigenstates of P have $P^2|q\rangle = M^2|q\rangle$ with $M^2 \geq (2m)^2$

→ continuum of multiparticle excitations, but no bound states (would have $M^2 \leq (2m)^2$)

to understand this better, consider full 2-point function

$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ (→ $i\Delta_F(x-y)$ in free case)

assume first $x_0 > y_0$:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|\phi(x)\,\mathbb{1}\,\phi(y)|\Omega\rangle$$

$$\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}|n_{\vec{p}}\rangle\langle n_{\vec{p}}|$$

$[H,\vec{P}]=0$, eigenstates of H are eigenstates of $\vec{P}$

define $|n_0\rangle$ as $\vec{P}|n_0\rangle = 0$, $H|n_0\rangle = m_n|n_0\rangle$

eigenstate of H with mom $\vec{p}\neq 0$ via boost $|n_{\vec{p}}\rangle = \Lambda_{\vec{p}}|n_0\rangle$

$$H|n_{\vec{p}}\rangle = E_p|n_{\vec{p}}\rangle = \sqrt{\vec{p}^2+m_n^2}\,|n_{\vec{p}}\rangle$$

→ $\sum_n$ sum over all 0-momentum states $|n_0\rangle$

$$\rightarrow \langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle \quad (\rightarrow 0)$$

$$+ \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} \underbrace{\langle\Omega|\phi(x)|n_{\vec{p}}\rangle}\langle n_{\vec{p}}|\phi(y)|\Omega\rangle$$

$$\langle\Omega|e^{iPx}\phi(0)e^{-iPx}|n_{\vec{p}}\rangle = e^{-ipx}\langle\Omega|\phi(0)|n_{\vec{p}}\rangle$$

$$= e^{-ipx}\langle\Omega|\underbrace{\Lambda_p^{-1}\phi(0)\Lambda_p}_{=\phi(0)\text{ ! scalar}}|n_0\rangle = e^{-ipx}\langle\Omega|\phi(0)|n_0\rangle$$

$$\rightarrow \langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip(x-y)} |\langle\Omega|\phi(0)|n_0\rangle|^2$$

$$\rightarrow \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\left(\Theta(x_0-y_0)e^{-ip(x-y)} + \Theta(y_0-x_0)e^{ip(x-y)}\right)|\cdots|^2$$

$$= \sum_n \int \frac{d^4p}{(2\pi)^4}\frac{i\,e^{-ip(x-y)}}{p^2-m_n^2+i0^+} |\langle\Omega|\phi(0)|n_0\rangle|^2$$

see definition & calculation of propagator, $\Delta_F(x-y)$ Section 4.2

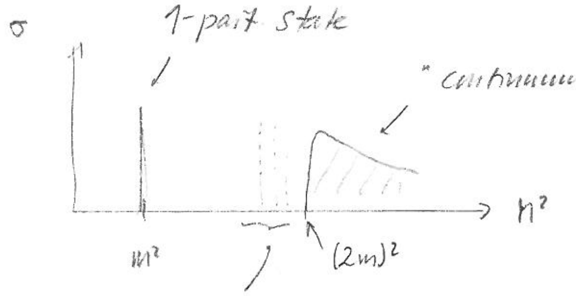
Källen-Lehmann spectral representation

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \int_0^\infty dM^2 \, \sigma(M^2) \, \Delta_F(x-y)\big|_{M^2}$$

spectral density $\sigma(M^2) = \sum_n \delta(M^2 - m_n^2) \, |\langle \Omega | \phi(0) | n_0 \rangle|^2$ $\hookrightarrow \otimes$

from our assumptions: $\sigma(M^2) = 0$ for $M^2 < m^2$

and $\sigma(M^2) = 0$ for $m^2 > M^2 > (2m)^2$



here would be bound states (which exist in many theories!)

$$\sigma(M^2) = Z \cdot \delta(M^2 - m^2) \quad + \quad \text{contributions for } M^2 > (2m)^2$$

field strength renormalization

exact mass (pole of 2 point function) need <u>not</u> be equal to parameter m_0 in $\mathcal{L}$

$$\langle \Omega | T \phi(x) \phi(0) | \Omega \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \left(\frac{iZ}{p^2 - m^2 + i0^+} + \int_{(2m)^2}^{\infty} dM^2 \, \frac{\sigma(M^2) \, i}{p^2 - M^2 + i0^+} \right)$$

single pole at $p^2 = m^2$ branch cut at $p^2 \geq (2m)^2$

if there are additional poles between $m^2 < p^2 < (2m)^2$
$\rightarrow$ bound states! (e.g. QED, positronium)

$\otimes$ note: in free case $\langle 0 | \phi | n \rangle \neq 0$ only for 1 particle state $|n\rangle$
in interacting case, $\langle 0 | \phi | n \rangle \neq 0$ also for "multi-particle" states

5.2 The S-matrix and the LSZ reduction formula

Typical scattering experiment (as in QM)

$\otimes$ $\begin{cases} \text{initial state, for } t \to -\infty \text{, described in terms of free fields } \phi_{in} \\ \text{final state, for } t \to +\infty \text{, described in terms of free fields } \phi_{out} \end{cases}$

$|\alpha_{in}\rangle$, as in free case, acting with creation operators on $|0\rangle$

$\langle \beta_{out}|$, again as in free case $\langle 0| a_{out} \ldots$

in & out states isomorphic → mapping S : $S|\alpha_{out}\rangle = |\alpha_{in}\rangle$

in terms of fields: $\phi_{out} = S^{-1} \phi_{in} S$ / S operator

S-matrix $S_{\beta\alpha} = \langle \beta_{out} | \alpha_{in}\rangle = \langle \beta_{out} | S | \alpha_{out}\rangle \equiv S_{\beta\alpha}$

S is unitary: $\delta_{\alpha\beta} = \langle \beta_{in} | \alpha_{in} \rangle = \langle \beta_{out} | \underbrace{S^{+} S}_{= \mathbb{1}} | \alpha_{out} \rangle$

$S = 1 + iT$ T: transition operator $S_{\beta\alpha} = \delta_{\beta\alpha} + i T_{\beta\alpha}$

Now consider $S_{\beta\alpha} \equiv \langle p_1 \ldots p_m, out | k_1 \ldots k_n ; in \rangle$

$$= \langle p_1 \ldots p_m ; out | (a^{+}_{k_1})_{in} | k_2 \ldots k_n ; in \rangle$$

$$(a_{k_1})_{in} = \int d^3\vec{x}\, e^{ik_1 x} \left(\omega_k \phi_{in}(x) + i \pi_{in}(x)\right) \qquad \text{(free fields!)}$$

$$= i \int d^3\vec{x}\, e^{ik_1 x} \overleftrightarrow{\partial_t} \phi_{in}(x) \qquad [\dot{\phi}_{in} = \pi_{in} !]$$

$$(a^{+}_{k_1})_{in} = -i \int d^3x\, e^{-ik_1 x} \overleftrightarrow{\partial_t} \phi_{in} \qquad \text{(also for in → out)}$$

We will also use
$$\left.\begin{aligned} \phi_{in}(x) &\to \frac{\phi(x)}{\sqrt{Z}} \quad \text{for } t \to -\infty \\ \phi_{out}(x) &\to \frac{\phi(x)}{\sqrt{Z}} \quad \text{for } t \to +\infty \end{aligned}\right\} \otimes$$

these relations do not hold as operator relations! only in weak sense

$\otimes$ basic assumption of scattering theory, often violated! (→ e.g. infrared problems)

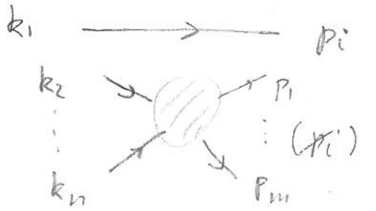
1st step in reduction

$$S_{\beta\alpha} = \langle p_1 .. p_m; out | (a^+_{k_1})_{in} | k_2 ... k_n; in\rangle$$

$$= \underbrace{\langle p_1 .. p_m; out | (a^+_{k_1})_{out} | k_2 ...\rangle} + \langle ... | \underbrace{(a^+_{k_1})_{in} - (a^+_{k_1})_{out}}_{S^c_{\beta\alpha} \ (\text{connected})} | ...\rangle$$

$$= \begin{cases} 0 & \text{if } p_i \neq k_1 \ \forall i \\ (2\pi)^3 \, 2\omega_{k_1} \, \delta(p_i - k_1) \, \langle p_1 .. \cancel{p_i} ... | k_2 ...\rangle \end{cases}$$

↳ disconnected contribution



$$S^c_{\beta\alpha} = -i \langle p_1 \ | \int d^3x_1 \, e^{-ik_1x_1} \overleftrightarrow{\partial}_{t_1} (\phi_{in}(x_1) - \phi_{out}(x_1)) | k_2 ...\rangle$$

$$= \frac{i}{\sqrt{Z}} \Big(\lim_{t_1 \to \infty} - \lim_{t_1 \to -\infty} \Big) \int d^3x_1 \, \underbrace{e^{-ik_1x_1} \overleftrightarrow{\partial}_{t_1} \langle p_1 ... | \phi(x_1) | k_2 ...\rangle}$$

$$= \frac{i}{\sqrt{Z}} \lim_{\substack{t_f \to \infty \\ t_i \to -\infty}} \int_{t_i}^{t_f} dt_1 \frac{\partial}{\partial t_1} \int d^3x_1 \quad - \text{"} - \text{"} - \text{"} -$$

$$= \frac{i}{\sqrt{Z}} \int_{-\infty}^{\infty} d^4x_1 \, \partial_{t_1} \Big(e^{-ik_1x_1} \overleftrightarrow{\partial}_{t_1} \langle p_1 ... | \phi(x_1) | k_2 ...\rangle \Big)$$

$$= \frac{i}{\sqrt{Z}} \int_{-\infty}^{\infty} d^4x_1 \Big(\underbrace{\big(-\partial^2_{t_1} e^{-ik_1x}\big)}_{= (-\nabla^2 + m) e^{-ik_1x_1}} \langle ... \rangle + e^{-ik_1x_1} \partial_t^2 \langle ... \rangle \Big)$$

↑ integrate by part

$$= \frac{i}{\sqrt{Z}} \int_{-\infty}^{\infty} d^4x_1 \, e^{-ik_1x_1} \big(\Box_{x_1} + m^2 \big) \langle p_1 ... p_m; out | \phi(x_1) | k_2 ... k_n; in\rangle$$

↑ $\partial_\mu \partial^\mu$ with $\partial^\mu = \frac{\partial}{\partial (x_1)_\mu}$

2nd step in reduction

$$\langle p_1 \dots p_m; out | \phi(x_1) | k_2 \dots k_n; in\rangle = \langle p_2 \dots p_m; out | (a_{p_1})_{out}\, \phi(x_1) | k_2 \dots k_n; in\rangle$$

$$= \underbrace{\langle \dots | \phi(x_1)(a_{p_1})_{in} | \dots \rangle}_{\text{disconnected}} + \langle \dots | \underbrace{(a_{p_1})_{out}\, \phi(x_1) - \phi(x_1)(a_{p_1})_{in}}_{i\int d^3y_1\, e^{ip_1y_1}\overleftrightarrow{\partial}_{t_1} \langle \dots | \phi_{out}(y_1)\phi(x_1) - \phi(x_1)\phi_{in}(y_1) | \dots \rangle} | \dots \rangle$$

from y_1, not x_1 ie $\partial_{t_1} = \frac{\partial}{\partial (y_1)^0}$

$$= \frac{i}{\sqrt{Z}}\left(\lim_{y_1^0 \to \infty} - \lim_{y_1^0 \to -\infty}\right) \int d^3y_1\, e^{ip_1y_1}\overleftrightarrow{\partial}_{t_1} \langle \dots | T\phi(x_1)\phi(y_1) | \dots \rangle$$

takes care of order in limit t → ±∞

proceed as before

$$S^c_{\beta\alpha} = \left(\frac{i}{\sqrt{Z}}\right)^2 \int d^4x_1\, e^{-ik_1x_1}(\Box_{x_1}+m^2) \int d^4y_1\, e^{ip_1y_1}(\Box_{y_1}+m^2)$$

$$\times \langle p_2 \dots p_m; out | T(\phi(x_1)\phi(y_1)) | k_2 \dots k_n; in\rangle$$

iterate procedure

$$iT_{\beta\alpha} = S^c_{\beta\alpha} = \left(\frac{i}{\sqrt{Z}}\right)^{n+m} \int d^4x_1\, e^{-ik_1x_1} \dots d^4x_n\, e^{-ik_nx_n}\, d^4y_1\, e^{ip_1y_1} \dots d^4y_m\, e^{ip_my_m}$$

$$\times (\Box_{x_1}+m^2) \dots (\Box_{y_m}+m^2) \langle \Omega | T(\phi(x_1) \dots \phi(x_n)\phi(y_1) \dots \phi(y_m)) | \Omega \rangle$$

LSZ reduction formula (Lehmann Symanzik & Zimmermann)

$\langle \Omega | T(\phi(x_1) \dots \phi(y_m)) | \Omega \rangle$ is a Green function of interacting theory

it has multipole structure

S matrix is (normalized) residue of multipole!

5.3. Perturbative expansion

How to compute $\langle \Omega | T \phi(x_1) \cdots \phi(x_n) | \Omega \rangle$

We need to express everything in terms of free fields and $|0\rangle$

↳ interaction picture

$$\underset{\uparrow}{\phi(t,\vec{x})} = e^{iH(t-t_0)}\underset{\nwarrow}{\phi(t_0,\vec{x})}e^{-iH(t-t_0)} = e^{iH\tau}\phi(t_0,\vec{x})e^{-iH\tau}$$

Schrödinger field, no t-dependence

$\tau \equiv t - t_0$

Heisenberg field

$$i\frac{\partial}{\partial t}\phi(t,\vec{x}) = [\phi(t,\vec{x}), H]$$

$$\phi(t,\vec{x}) = e^{iH\tau}e^{-iH_0\tau}\underbrace{e^{iH_0\tau}\phi(t_0,\vec{x})e^{-iH_0\tau}}_{\equiv\ \phi_I(t,\vec{x})}e^{iH_0\tau}e^{-iH\tau}$$

interaction picture

$H = H_0 + H_{int}$: H_0 : free Hamiltonian

$$\phi(t,\vec{x}) = U^{+}(t,t_0)\,\Phi_I(t,\vec{x})\,U(t,t_0)$$

(evolution operator $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$

Note: Φ_I evolves with H_0 : $i\frac{\partial}{\partial t}\Phi_I(t,\vec{x}) = [\Phi_I, H_0]$ free field

Show: • $i\frac{\partial}{\partial t}U(t,t_0) = H_I(t)\,U(t,t_0)$

interaction Hamiltonian in interaction picture $H_I = e^{iH_0(t-t_0)}H_{int}\,e^{-iH_0(t-t_0)}$

- formal solution

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right)$$

(as in QM)

- $U^{+}(t,t_0) = U(t_0,t)$ and $U(t_1,t_0)\,U(t_0,t_2) = U(t_1,t_2)$

→ can express $\phi(x_i)$ in terms of free fields $\phi_I(x_i)$

now assume $t_1 > t_2 > \ldots t_n$ and compute

$\langle\Omega|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle$

$$= \langle\Omega|\underbrace{U^+(t_1,t_0)}_{U(t_0,t_1) = U(t_0,T)U(T,t_1) = U^+(T,t_0)U(T,t_1)}\phi_I(x_1)\underbrace{U(t_1,t_0)\cdot U^+(t_2,t_0)}_{= U(t_1,t_0)U(t_0,t_2) = U(t_1,t_2)}\phi_I(x_2)U(t_2,t_0)\cdots U^+(t_n,t_0)\phi_I(x_n)\underbrace{U(t_n,t_0)}_{U(t_n,-T)U(-T,t_0)}|\Omega\rangle$$

$$= \langle\Omega|U^+(T,t_0)\Big(U(T,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)\cdots\phi_I(x_n)U(t_n,-T)\Big)U(-T,t_0)|\Omega\rangle$$

$$= \langle\Omega|U^+(T,t_0)\,T\Big(\phi_I(x_1)\cdots\phi_I(x_n)U(T,-T)\Big)U(-T,t_0)|\Omega\rangle$$

assume $T > t_1$ and $-T < t_n$ (i.e. $T \to \infty$)

Note t_0 is arbitrary reference, set $t_0 \to 0$ for notational simplicity

Next need to relate $|\Omega\rangle \sim |0\rangle$ and consider $U(-T,0)|\Omega\rangle$ with limit $T\to\infty$ in mind

consider $e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$

$E_n > E_0$ (by def. of $|\Omega\rangle$)

(note: E_0 not necessarily 0, since 0-level defined by $H_0|0\rangle = 0$)

taking lim $T\to\infty$ (including factor $-i0^+$ to ensure convergence) only 1st term survives!

$$\lim_{T\to\infty(1-i0^+)} e^{-iHT}|0\rangle = \lim_{T\to\infty(1-i0^+)} e^{-iE_0T}\langle\Omega|0\rangle\cdot|\Omega\rangle$$

$$\Rightarrow |\Omega\rangle = \lim_{T\to\infty(1-i0^+)}\frac{e^{-iHT}|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle}$$

$U(-T,0) = e^{-iH_0T}e^{iHT}$ and $e^{-iH_0T}|0\rangle = |0\rangle$

$$\Rightarrow \lim_{T\to\infty(1-i0^+)} U(-T,t_0=0)|\Omega\rangle = \lim_{T\to\infty(1-i0^+)}\frac{|0\rangle}{e^{-iE_0T}\langle\Omega|0\rangle}$$

similar: $\lim_{T\to\infty} \langle\Omega| U^+(T, t_0=0) = \lim_{T\to\infty} \frac{\langle 0|}{e^{-iE_0T}\langle 0|\Omega\rangle}$

and write denominator as

$$e^{-iE_0T}\langle 0|\Omega\rangle\, e^{-iE_0T}\langle\Omega|0\rangle = \langle 0|\underbrace{e^{iH_0T}e^{-iHT}}_{U(T,0)}|\Omega\rangle\langle\Omega|\underbrace{e^{-iHT}e^{iH_0T}}_{U^+(-T,0)}|0\rangle$$

$$= \langle 0|U(T,-T)|0\rangle$$

$$\Rightarrow \boxed{\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \frac{\langle 0|T\left(\phi_I(x_1)\cdots\phi_I(x_n)\, e^{-i\int_{-\infty}^{\infty}d\tau\, H_I(\tau)}\right)|0\rangle}{\langle 0|T\left(e^{-i\int_{-\infty}^{\infty}d\tau\, H_I(\tau)}\right)|0\rangle}}$$

RHS is expressed entirely in terms of free fields (at the price of having infinitely many terms due to exponential)

→ Gell-Mann Low formula

$$g(x_1\ldots x_N) = \frac{\langle 0|T(\phi(x_1)\cdots\phi(x_n)\, e^{i\int d^4x\, \mathcal{L}_{int}(\phi)})|0\rangle}{\langle 0|T\, e^{i\int d^4x\, \mathcal{L}_{int}(\phi)}|0\rangle}$$

For ϕ^4 e.g. $\int_{-\infty}^{\infty}d\tau\, H_I(\tau) = \int d^4x\, \mathcal{H}_I = \int d^4x \frac{\lambda_0}{4!}\phi_I^4(x)$

↗

$$H_I(t) = e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)} = e^{iH_0(t-t_0)}\int d^3\vec{x}\, \frac{\lambda_0}{4!}\phi^4(t_0,\vec{x})\, e^{-iH_0(t-t_0)}$$

$$= \int d^3\vec{x}\, \frac{\lambda_0}{4!}\phi_I^4(t,\vec{x})$$

If H_I is "small" (coupling λ_0 ~ small) can use perturbation theory: expand exponential and keep only first few terms

→ need to evaluate $\langle 0|T(\underbrace{\phi_I(x_1)\cdots\phi_I(x_m)}_{})|0\rangle$

from $\phi_I(x_1)\cdots\phi_I(x_n)$ and expansion of $e^{-i\int_{-\infty}^{\infty}d^4x\, \mathcal{H}_I}$

5.4. Wick's theorem and Feynman diagrams

Notation: drop subscript I, all fields understood in interaction picture

To evaluate $\langle 0|T(\phi_I(x_1)\dots\phi_I(x_n))|0\rangle \xrightarrow{\text{notation}} \langle 0|T(\phi(x_1)\dots\phi(x_n))|0\rangle$

write $\phi(x) = \phi_+(x) + \phi_-(x) = \int d\tilde{p}\, a_{\vec p}\, e^{-ipx} + \int d\tilde{p}\, a^+_{\vec p}\, e^{ipx}$

can do this, since $\phi\ (\equiv \phi_I)$ is free field!

Now
$$T(\phi(x)\phi(y)) = \theta(x_0-y_0)\left(\phi^+(x)\phi^+(y) + \underbrace{\phi^+(x)\phi^-(y)} + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y)\right)$$
$$+\ \theta(y_0-x_0)\left(\phi^+(y)\phi^+(x) + \underbrace{\phi^+(y)\phi^-(x)} + \phi^-(y)\phi^+(x) + \phi^-(y)\phi^-(x)\right)$$

$$\phi^-(y)\phi^+(x) + [\phi^+(x)\phi^-(y)] \qquad = \phi^-(x)\phi^+(y) + [\phi^+(y)\phi^-(x)]$$

all other terms are normal ordered

$$\to\ T(\phi(x)\phi(y)) = :\phi(x)\phi(y): + \underbracket{\phi(x)\phi(y)}$$

contraction of two fields

vanishes if sandwiched between $\langle 0|\dots|0\rangle$

$$|||$$

$$\theta(x_0-y_0)\,[\phi^+(x),\phi^-(y)] + \theta(y_0-x_0)\,[\phi^+(y)\phi^-(x)]$$
$$= i\Delta_F(x-y) = g(x,y)$$

Consider now 4 fields:

$$T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) = T(\phi_1\phi_2\phi_3\phi_4)$$
$$= :\phi_1\phi_2\phi_3\phi_4: + :\phi_1\phi_2:\underbracket{\phi_3\phi_4} + :\phi_1\phi_3:\underbracket{\phi_2\phi_4} + \dots$$

4 more single contractions

$$+ \underbracket{\phi_1\phi_2}\ \underbracket{\phi_3\phi_4} + \underbracket{\phi_1\phi_3}\ \underbracket{\phi_2\phi_4} + \underbracket{\phi_1\phi_4}\ \underbracket{\phi_2\phi_3}$$

or in more compact notation

$$T(\phi_1\phi_2\phi_3\phi_4) = :\ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\overbracket{\phi_3\phi_4} + \phi_1\overbracket{\phi_2\phi_3\phi_4} + \dots$$
$$+ \underbracket{\phi_1\phi_2}\,\underbracket{\phi_3\phi_4} + \wick{\c2{\phi_1}\c1{\phi_2}\c2{\phi_3}\c1{\phi_4}} + \underbracket{\phi_1\overbracket{\phi_2\phi_3}\phi_4}\ :$$

For $\langle 0|T(\cdots)|0\rangle$ only fully contracted contributions survive

$\langle 0|T(\phi_1\cdots\phi_{2n+1})|0\rangle = 0!$ (odd number of fields)

Wick's theorem

$$T\,(\phi(x_1)\cdots\phi(x_n)) \;=\; :\phi(x_1)\cdots\phi(x_n) \;+\; \text{all possible contractions}:$$

proof by induction (→ exercise)

Use Wick's theorem to compute 4 point green functions perturbatively

Start with $\langle 0|T(\phi_1\phi_2\phi_3\phi_4)|0\rangle$

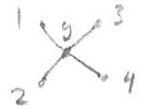
$$= \underbrace{\phi_1\phi_2}\,\underbrace{\phi_3\phi_4} + \underbrace{\phi_1\overbrace{\phi_2\phi_3}}\phi_4 + \underbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4}$$

$$= (i\Delta_F(x_1-x_2)\,(i\Delta_F(x_3-x_4)) + i\Delta_F(x_1-x_3)\,(i\Delta_F(x_2-x_4)) + (i\Delta_F(x_1-x_4))\,(i\Delta_F(x_2-x_3))$$

$\simeq$   

next $\int d^4y\,\langle 0|T(\phi_1\phi_2\phi_3\phi_4\,\frac{-i\lambda_0}{4!}\,\phi_y^4)|0\rangle$

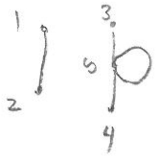
$$= \underbrace{\phi_1\,\underbrace{\phi_2\,\overbrace{\phi_3\,\overbrace{\phi_4\;\;\phi_y^4}}^{2}}_{3}}_{4}\cdot\frac{-i\lambda_0}{4!} \qquad 4!\ \text{possibilities!}$$



$$= \int d^4y\,(i\Delta_F(x_1-y))\,(i\Delta_F(x_2-y))\,(i\Delta_F(x_3-y))\,(i\Delta_F(x_4-y))\,(-i\lambda_0)$$

connected contribution

$$+ \;\underbrace{\phi_1\phi_2}\;\;\underbrace{\phi_3\,\underbrace{\phi_4\;\;\overbrace{\phi_y^4}}_{3}}_{4}\;\frac{-i\lambda_0}{4!} \qquad 4\cdot3\ \text{possibilities}$$

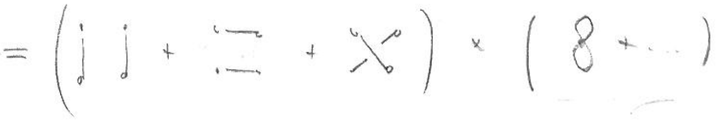


$$= (i\Delta_F(x_1-x_2))\int d^4y\,(i\Delta_F(x_3-y))\,(i\Delta_F(x_4-y))\cdot\frac{-i\lambda_0}{2} \qquad \left(\tfrac{1}{2}\ \text{symmetry factor}\right)$$

with further permutations $1\leftrightarrow2\leftrightarrow3\leftrightarrow4$: disconnected contribution

$$+ \;\underbrace{\phi_1\phi_2}\;\;\underbrace{\phi_3\phi_4}\;\;\underbrace{\overbrace{\phi_y^4}}\;\frac{-i\lambda_0}{4!} \qquad 3\ \text{possibilities}$$

$\left(\frac{-i\lambda_0}{8}\right)$



vacuum to vacuum transitions $\langle 0|e^{-i\int d^4y\,H_I}|0\rangle$

The vacuum bubbles (vacuum to vacuum transitions) generated in the numerator of

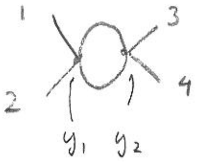
$$g(x_1, \dots x_n) \equiv \frac{\langle 0 | T \phi(x_1) \cdots \phi(x_n)\, e^{-i\int H_I d^4y} |0\rangle}{\langle 0 | e^{-i\int H_I d^4y} |0\rangle}$$

are divided out by denominator!

→ Feynman rules for Green function in coordinate space:

- draw all distinct diagrams with external labels $x_1 \dots x_n$
- associate a propagator $i\Delta_F(x-y)$ with each line connecting x and y
- associate $-i\lambda_0 \int d^4y$ to each vertex at y
- divide diagram by symmetry factor S

distinct diagram : topologically different, after labelling external legs



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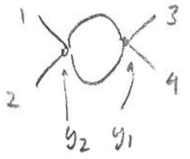

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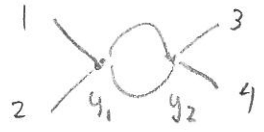
symmetry factor : count number of contractions leading to same diagram!

e.g.



$$\underbrace{\phi_1 \overbrace{\phi_2 \phi^4_{y_1}}^{3}}_{4} \; \underbrace{\overbrace{\phi^4_{y_2} \phi_3 \phi_4}^{3}}_{4} \qquad \left(\frac{-i\lambda_0}{4!}\right)^2$$

$$\underbrace{\underbrace{\phi_{y_1}\phi_{y_2}}_{2}}_{1}$$

$$\rightarrow S = \frac{4\cdot 3\cdot 4\cdot 3\cdot 2}{(4!)^2} = \frac{1}{2}$$

factor 2 from exchanging $y_1 \leftrightarrow y_2$ cancelled by $\frac{1}{2}$ from expansion of $\exp(-i\int d^4x\, \mathcal{L})$

5.5. Computing S-matrix elements via Feynman rules

Let us now make connection Feynman diagrams <=> S matrix

It's easier to work in momentum space.

Consider again $g(x_1, x_2, x_3, x_4) = \int \prod_{i=1}^{4} \frac{d^4 p_i}{(2\pi)^4} e^{-ix_i p_i} g(p_1, p_2 p_3 p_4)$

make Fourier trsf of LSZ reduction formula, using

$$(\Box_{x_j} + m^2)\, g(x_1 \dots x_n) = \int \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4} e^{-ix_i p_i} (-p_j^2 + m^2)\, g(p_1 \dots p_n)$$

We get (see page 61)

$$\prod_{i=1}^{m} \left(\frac{i\sqrt{Z}}{p_i^2 - m^2}\right) \prod_{j=1}^{n} \left(\frac{i\sqrt{Z}}{k_i^2 - m^2}\right) \underbrace{(iT_{\beta\alpha})}_{\langle p_1 \;\; p_m | iT | k_1 \dots k_n\rangle} = \int \prod_{i=1}^{m} (d^4 y_i\, e^{ip_i y_i}) \prod_{j=1}^{n} (d^4 x_j\, e^{-ik_j x_j}) \times$$

$$\langle \Omega | T(\phi(x_1) \dots \phi(x_n) \phi(y_1) \dots \phi(y_m)) | \Omega \rangle$$

compute 4-point function to order $(\lambda_0)^0$

$$\int d^4x_1 d^4x_2 d^4y_1 d^4y_2\, e^{i(p_1 y_1 + p_2 y_2 - k_1 x_1 - k_2 x_2)} \underbrace{\langle 0 | T(\phi_I(x_1) \phi_I(x_2) \phi_I(y_1) \phi_I(y_2)) | 0 \rangle}_{= i\Delta_F(x_1 - x_2)\, i\Delta_F(y_1 - y_2) \; + 2 \text{ more terms}}$$

$$\int d^4x_1 \dots d^4y_2\; e^{ip_1(y_1 - y_2)} e^{iy_2(p_1 + p_2)} e^{-ik_1(x_1 - x_2)} e^{-i(k_1 + k_2)x_2} \left(i\Delta_F(x_1 - x_2)\, i\Delta_F(y_1 - y_2) + \dots \right)$$

shift $y_1 \to y_1 + y_2 \quad x_1 \to x_1 + x_2$

$$\int d^4y_2\, e^{iy_2(p_1 + p_2)} \int d^4x_2\, e^{ix_2(k_1 + k_2)} \int d^4y_1\, e^{ip_1 y_1} i\Delta_F(y_1) \int d^4x_1\, e^{-ik_1 x_1} i\Delta_F(x_1)$$

$$(2\pi)^4 \delta(p_1 + p_2)\; (2\pi)^4 \delta(k_1 + k_2)\; \frac{i}{p_1^2 - m_0^2 + i0^+} \;\; \frac{i}{k_1^2 - m_0^2 + i0^+}$$

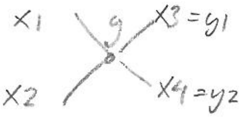
(recall $\Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ipx}}{p^2 - m_0^2 + i0^+}$ → prop in momentum space $\frac{i}{p^2 - m_0^2 + i0^+}$

This contribution to the Green function has poles @

$p_1^2 = m_0^2$ and $k_2^2 = m_0^2$, (2 poles)

but does not contribute to $\langle p_1 p_2 | iT | k_1 k_2 \rangle$: need 4 poles!

only connected diagrams can have n poles (for n-point fct.)

Consider (again) contribution  and take Fourier trsf.

$$\int d^4x_1 \dots d^4y_2 \, e^{i(p_1y_1 + p_2y_2 - k_1x_1 - k_2x_2)} \int d^4y \; i\Delta_F(x_1 - y) \, i\Delta_F(x_2 - y) \, i\Delta_F(y_1 - y) \, i\Delta_F(y_2 - y) (-i\lambda$$

$x_1 \to x_1 + y$ etc.

$$\int d^4y \, e^{iy(p_1+p_2-k_1-k_2)} \; (-i\lambda_0) \int d^4x_1 \, i\Delta_F(x_1) \; \cdots \; \int d^4y_2 \, i\Delta_F(y_2)$$

$$= (2\pi)^4 \delta(p_1+p_2-k_1-k_2) \; (-i\lambda_0) \frac{i}{k_1^2 - m_0^2 + i0+} \; \cdots \cdots \; \frac{i}{p_2^2 - m_0^2 + i0+}$$

Feynman diagram for Green function in momentum space

propagator $\frac{i}{p^2 - m_0^2 + i0^+}$

To obtain S-matrix, take residue of multiparticle pole (i.e. cut-off external legs or amputate graph)

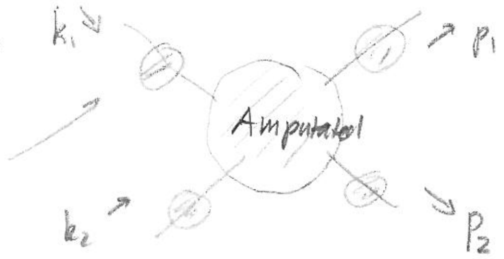
assuming for the moment $Z = 1$ and $m_0^2 = m^2$

$$\langle p_1 p_2 | iT | k_1 k_2 \rangle \equiv (2\pi)^4 \delta(p_1+p_2-k_1-k_2) \, i\mathcal{M} = -i\lambda_0$$

def of invariant amplitude $\mathcal{M}$

Let's look at "amputation" more carefully

structure of 4point <u>Green</u> function:

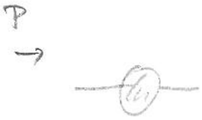


1 particle irreducible self-energy

↳ cannot be cut into 2 by cutting 1 line only

for ϕ^4: @ $(\lambda_0)^2$:  is 1PI → 

 is <u>not</u> 1PI → 



$$= \frac{i}{p^2-m_0^2} + \frac{i}{p^2-m_0^2}\left(-i\Pi(p^2)\right)\frac{i}{p^2-m_0^2} + \dots$$

↑

$\equiv$  note $\Pi(p^2) \sim O(\lambda_0)$

$$= \frac{i}{\underbrace{p^2-m_0^2-\Pi(p^2)}} \xrightarrow[\text{pole!}]{\text{expand around}} \frac{iZ}{p^2-m^2} + \text{regular}$$

position of pole is shifted from m_0^2 (shift $O(\lambda_0)$!)

at order $(\lambda_0)^0$: $Z=1$, $m_0^2=m^2$

from Green function to scattering matrix element:

multiply by $\left(\frac{i\sqrt{Z}}{p^2-m^2}\right)^{-1}$ for each external leg!

$\Rightarrow \langle p_1,p_2|\,iT\,|k_1,k_2\rangle = (\sqrt{Z})^4 \cdot$ 

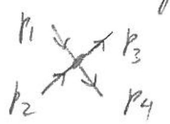
Feynman rules for scattering matrix elements in momentum space

- draw all connected and amputated graphs

  no external leg can be cut-off any longer by cutting only a single line

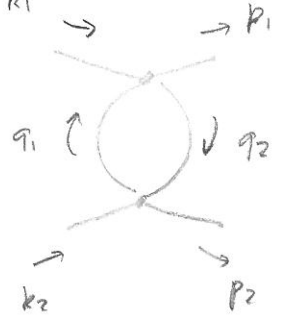
- (for each external line add factor 1)
- for each internal line (propagator) $\frac{i}{p^2 - m_0^2 + i0^+}$

- for each vertex $-i\lambda_0$



- integrate over all internal momenta $\int \frac{d^4p}{(2\pi)^4}$
- impose momentum conservation at each vertex $(2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)$
- add symmetry factor

example.



$$\int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \overset{(2\pi)^4}{\delta}(k_1 + q_1 - p_1 - q_2) \overset{(2\pi)^4}{\delta}(k_2 + q_2 - q_1 - p_2)$$

$$(-i\lambda_0)^2 \frac{i}{q_1^2 - m_0^2 + i0^+} \frac{i}{q_2^2 - m_0^2 + i0^+} \times \frac{1}{2}$$

symmetry factor

↓ do q_2 integration, $q_2 = k_1 + q_1 - p_1$

$$\underbrace{(2\pi)^4 \delta(k_1 + k_2 - p_1 - p_2)}_{\text{overall momentum conservation}} \frac{(-i\lambda_0)^2}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{i}{q_1^2 - m_0^2 + i0^+} \frac{i}{(q_1 + k_1 - p_1)^2 - m_0^2 + i0^+}$$

symmetry factor

1-loop diagram → 1 undetermined momentum → 1-loop integral

$$iT = iM \cdot (2\pi)^4 \delta(\Sigma p_{in} - \Sigma p_{out})$$

δ-function always present 4-mom conservation