QFT I HS 2013

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Books:
- Quantum Field Theory, a modern introduction — M. Kaku, Oxford U.P.
- An introduction to QFT — Peskin Schroeder
- QFT — Ryder, Cambridge U.P.
- QFT — Itzykson Zuber (not for first reading!)
- The quantum theory of fields — S. Weinberg, Vol I

Outlook:
- Combine relativity and QM
- Single particle rel. QM has severe limitations
  ↳ Quantum Field Theory
    ↳ describes processes where number of particles change
- here canonical quantization of field theory
  ↳ no path integrals
    ↳ no non-Abelian theories
- Scattering processes, loop correction, renormalization

Notation & conventions

QM + Relativity

<u>natural units</u> $\hbar = c = 1$

→ Velocity : pure number

$[\text{energy}] = [\text{mass}] = [\text{momentum}] = [\text{length}]^{-1}$

$\hbar c = 200 \text{ MeV fm} \Rightarrow 200 \text{ MeV} = 1 \text{ fm}^{-1}$

<u>Lorentz indices</u> $\mu, \nu, (\rho, \sigma \ldots) \in \{0, 1, 2, 3\}$

metric: $g^{\mu\nu} = \text{Diag}(1, -1, -1, -1)$

$x^\mu = (x^0, x^i) = (t, \vec{x})$ $\qquad \partial_\mu = (\frac{\partial}{\partial x^0}, \vec{\nabla}) = (\partial_t, \vec{\nabla})$

Spatial indices $i, j, k \ldots \in \{1, 2, 3\}$

$x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{x})$

totally antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$; $\varepsilon^{0123} = +1 = -\varepsilon_{0123}$

momentum operator $p^\mu = i\partial^\mu \Rightarrow \begin{cases} E = i\partial_t \\ \vec{p} = -i\vec{\nabla} \end{cases}$

Lorentz trsf: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$: $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$ $\quad$ 2→

6 parameters $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (3 rotations, 3 boosts)

<u>Spinor indices</u> $\alpha, \beta\ (\gamma, \ldots \delta \ldots) \in \{1, 2, (3, 4)\}$

Weyl spinor

Dirac spinor

1 Classical Field Theory

1.1. The action principle

recall classical mechanics in Lagrangian formalism: $i = 1, \dots N$ dof

coordinates $q_i(t)$ $\rightarrow$ Lagrangian $L(q_i, \dot{q}_i)$

action $S = \int dt \, L(q_i, \dot{q}_i)$

fix $q_i(t_{in})$ and $q_i(t_f)$, then classical trajectory is extremum of S, i.e.

$$\delta S = \delta \int_{t_{in}}^{t_f} dt \, L(q_i, \dot{q}_i) = 0$$

$$0 = \delta S = \int_{t_{in}}^{t_f} dt \sum_i \Big(\frac{\partial L}{\partial q_i} \delta q_i + \underbrace{\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i}_{\frac{\partial L}{\partial \dot{q}_i} \frac{\partial}{\partial t} \delta q_i \; = \; -\left(\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i} \Big) = \int_{t_{in}}^{t_f} dt \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i$$

no boundary term! $\delta q_i(t_{in}) = \delta q_i(t_f) = 0$!

true $\forall \, \delta q_i$ $\rightarrow$ Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{q}_i} = 0$$

Hamiltonian formalism: define conjugate momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$H(p_i, q_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i) \quad \text{with} \quad \dot{q}_i = \dot{q}_i(q, p)$$

consider now a field $\phi(t, \vec{x}) \simeq \{q_i(t)\}$

$\uparrow$ continuous "label" i, $N \to \infty$

$$L(q_i(t), \dot{q}_i(t)) \to L(\phi(t,\vec{x}), \partial_\mu \phi(t,\vec{x}))$$

$\uparrow$ Lorentz covariance

$$S = \int dt\, L(\phi, \partial_\mu \phi) = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

$$L = \int d^3x\, \mathcal{L}(\phi, \partial_\mu \phi)$$

Lagrangian — Lagrangian density — language often sloppy! — 'drop'

locality (and Lorentz covariance):

The Lagrangian (density) depends only on ϕ and $\partial_\mu \phi$

("nearest neighbour" interaction)

<u>Equation of motions</u> (Euler Lagrange) from action principle

$$0 = \delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}_{} \right)$$

Surface term vanishes, boundary conditions $\left(\phi(\vec{x} \to \infty) \to 0 \right.$ sufficiently fas

$$\to \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

Note ϕ and $\partial_\mu \phi$ are considered independent!

Eq. of motion describe evolution of field configuration

$\phi(x)$ from t_{in} to t_f

<u>Note:</u> changing $\mathcal{L} \to \mathcal{L} + \underbrace{\partial_\mu K^\mu}_{\text{total derivative}}$ leaves EoM unchanged

since S is not affected

(always assuming that boundary terms vanish)

Lagrangian formulation of field theory well suited for relativistic theories (covariant!)

Hamiltonian formalism mainly to make contact with QM

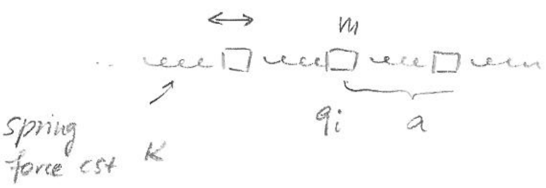
$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} \quad \text{conjugate momentum fields}$$

$$H = \int d^3x \, (\pi \dot{\phi} - \mathcal{L}) \equiv \int d^3x \, \mathcal{H}$$

Hamiltonian (density)

<u>Example</u> <u>1+1 dimension</u> non-relativistic system

<u>continuum</u>



<u>discrete</u> q_i, $i = 1..N$

$$L = T - V = \sum_i \frac{m}{2} \dot{q}_i^2 - \frac{K}{2} (q_{i+1} - q_i)^2$$

↳ exercise

$\longrightarrow$ rod, mass density $\mu = \lim_{a \to 0} \frac{m}{a}$ and Youngs modulus $Y = \lim_{a \to 0} K \cdot a$

$q_i(t) \to \phi(t, x)$

$q_{i+1}(t) \to \phi(t, x+a)$

continuum limit

Euler-Lagrange

$$0 = m \ddot{q}_i + K(q_i - q_{i-1}) - K(q_{i+1} - q_i)$$

$$\mu \frac{\partial^2 \phi(t,x)}{\partial t^2} - Y \frac{\partial^2 \phi(t,x)}{\partial x^2} = 0$$

$$\to \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}\right)$$

<u>Example</u>: real scalar field (Klein-Gordon) ϕ

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2}_{\text{free}} - \underbrace{\frac{\lambda}{4!}\phi^4}_{\text{interactions}} \simeq -\frac{1}{2}\phi\,\partial_\mu\partial^\mu\phi - \frac{m^2}{2}\phi^2$$

$$\left.\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= \partial^\mu\phi \\ \frac{\partial\mathcal{L}}{\partial\phi} &= -m^2\phi - \frac{\lambda}{3!}\phi^3 \end{aligned}\right\} \quad \text{EoM: } \underbrace{\partial_\mu\partial^\mu\phi + m^2\phi}_{\text{free Klein-Gordon equation}} + \frac{\lambda}{3!}\phi^3 = 0$$

solution $\phi(x) = A\,e^{ip^\mu x_\mu}$ with $p^2 = m^2$

<u>Example</u>: complex scalar field

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\partial^\mu\phi^* + m^2\phi^* = 0$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} - \frac{\partial\mathcal{L}}{\partial\phi^*} = \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

1.2 Noether's Theorem

So far, no statement about $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ (i not spatial index)

Consider now the case, where there is a continuous symmetry, i.e. under a certain transformation the EoM are unchanged

$$\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu$$

$\Rightarrow$ this will result in a conserved quantity!

- internal symmetries $\phi_i \to \phi_i' = \phi_i + \Delta\phi_i$
- space-time symmetries $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

} cont. linked to $\mathbb{1}$

Internal symmetry (transformation of fields)

Assume that under $\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \Delta_i\phi(x)$ we have $\mathcal{L} \to \mathcal{L} + \partial_\mu K^\mu \equiv \mathcal{L} + \Delta\mathcal{L}$, i.e. change of $\mathcal{L}$ is total derivative

compute $\Delta\mathcal{L}$ explicitly

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_i}\Delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Delta(\partial_\mu\phi_i)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Delta\phi_i\right) + \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)}_{=0 \ \text{EoM}}\Delta\phi_i \overset{!}{=} \partial_\mu K^\mu$$

$\Rightarrow$ conserved current $\boxed{\partial_\mu j^\mu = 0}$ with $\boxed{j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Delta\phi_i - K^\mu}$

only if ϕ are solutions to EoM

$\Rightarrow$ conserved "charge" $Q = \int d^3\vec{x}\, j^0(x)$

here conserved means $\frac{d}{dt}Q = 0$

$$\left(\dot{Q} = \int d^3\vec{x}\, \partial^0 j^0 \xrightarrow{\partial_\mu j^\mu = 0} = -\int d^3\vec{x}\, \vec{\nabla}\cdot\vec{j} = 0,\right.$$ assuming usual boundary conditions for $x \to \infty$)

Space-time symmetries

Consider translation $\quad x^\mu \to x'^\mu = x^\mu + a^\mu \qquad \delta x^\mu = a^\mu$

$$\to \quad \Delta\phi = \phi(x+a) - \phi(x) = a^\nu \partial_\nu \phi(x)$$

$$\to \quad \Delta\mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}} a^\nu \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} a^\nu \partial_\nu \partial_\mu \phi = \underbrace{a^\mu \partial_\mu \mathcal{L}}_{a^\nu \partial_\mu g^\mu{}_\nu \mathcal{L}}$$

$$a^\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right) = a^\nu \partial_\mu g^\mu{}_\nu \mathcal{L}$$

This holds $\forall a^\nu$, thus

$$\partial_\mu T^\mu{}_\nu = 0 \qquad \text{with} \qquad T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - g^\mu{}_\nu \mathcal{L}$$

$T^{\mu\nu}$ is called the energy-momentum tensor and corresponds to four ($\nu \in \{0,1,2,3\}$) conserved currents P^μ (we actually considered four symmetries, shift a^0, a^i)

$$P^\mu = \int d^3\vec{x}\; T^\mu{}_0 \qquad \text{with} \qquad \frac{d}{dt} P^\mu = 0$$

<u>Note</u>: $\quad P^0 = \int d^3\vec{x}\; T^0{}_0 = \int d^3\vec{x} \left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \right) = \int d^3\vec{x} \left(\pi \dot\phi - \mathcal{L} \right) = H \quad !$

<u>Note</u>: $\quad T^{\mu\nu}$ is not unique, can add term $\partial_\rho E^{\rho\mu\nu}$ with antisymmetric $E^{\rho\mu\nu} = -E^{\mu\rho\nu}$

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho E^{\rho\mu\nu} \quad \text{also conserved since } \partial_\mu \partial_\rho E^{\rho\mu\nu} = 0$$

this freedom can be used to make $T^{\mu\nu}$ symmetric $(= T^{\nu\mu})$

<u>Exercise</u> $\quad$ Consider LT $\quad \delta x^\mu = \omega^\mu{}_\nu x^\nu$

$\to$ conserved currents $\mathcal{M}^{\rho\mu\nu} \equiv T^{\rho\nu} x^\mu - T^{\rho\mu} x^\nu, \quad \partial_\rho \mathcal{M}^{\rho\mu\nu} = 0$

$\to$ conserved "charges" $\quad M^{\mu\nu} = \int d^3\vec{x}\; \mathcal{M}^{0\mu\nu}, \qquad \dot{M}^{\mu\nu} = 0$

1.3 The free Klein-Gordon field

Consider a real $\underbrace{\text{scalar}}$ field $\phi(x) = \phi(t, \vec{x})$

scalar: under LT $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$: $\phi(x) \to \phi'(x') = \phi(x)$

To obtain Linv theory, want $S = S'$ (form invariance of EoM)

$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$ invariant if $\mathcal{L}(\phi, \partial_\mu\phi)$ invariant

$\to$ $\boxed{\mathcal{L} = \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \phi^2}$ $\to$ Linv theory

$\hookrightarrow$ EoM : Klein-Gordon eq. $\boxed{\begin{aligned} &\partial_\mu \partial^\mu \phi + m^2 \phi = 0 \\ \text{or } &(\Box + m^2)\phi = 0 \end{aligned}}$

general solution (most general lin. comb of sol's $A \cdot e^{\pm ipx}$ with $p^2 = m^2$)

$$\phi(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p}}_{\text{Linv}} \left(a_p e^{-ipx} + a_p^* e^{+ipx} \right)$$

coefficients

$px = p^\mu x_\mu = \omega_p t - \vec{p} \cdot \vec{x}$

$p^2 = m^2 \Rightarrow \omega_p = \sqrt{\vec{p}^{\,2} + m^2}$

$$\int \frac{d^4p}{(2\pi)^4} (2\pi) \, \delta(p^2 - m^2) \, \Theta(p^0) \quad \text{is manifestly Linv}$$

$(d^4p' = |\det \Lambda| \, d^4p = d^4p)$

$$= \int \frac{d^4p}{(2\pi)^3} \delta(p_0^2 - \omega_p^2) \, \Theta(p^0)$$

use $\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x_i)$ with $f(x_i) = 0$

$$= \int \frac{d^4p}{(2\pi)^3} \frac{1}{2\omega_p} \left(\delta(p_0 + \omega_p) + \delta(p_0 - \omega_p) \right) \Theta(p^0)$$

no contribution since $\omega_p > 0 \leftrightarrow \Theta(p^0)$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2\omega_p}$$

Conjugate momentum field $\quad \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$

$$\mathcal{L} = \frac{1}{2}\dot\phi^2 - \frac{1}{2}(\vec\nabla\phi)^2 - \frac{m^2}{2}\phi^2$$

$$\mathcal{H} = \pi\dot\phi - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{m^2}{2}\phi^2$$

To get a better idea what this "theory" is, compute $H = \int d^3x\, \mathcal{H}$

$$\phi = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}}\left(a_p e^{-i\omega t + i\vec p\vec x} + a_p^* e^{i\omega t - i\vec p \vec x}\right)$$

$$\dot\phi = \int \widetilde{dp}\,\left(-i\omega_p a_p e^{-ipx} + i\omega_p a_p^* e^{ipx}\right)$$

$$\frac{1}{2}\dot\phi^2 = \frac{1}{2}\int\widetilde{dp}\int\widetilde{dp}'\; i^2\Big(\omega\omega' a a' e^{-i(\omega+\omega')t} e^{i(\vec p+\vec p')\vec x} - \omega\omega' a a'^* e^{-i(\omega-\omega')t} e^{i(\vec p-\vec p')\vec x}$$

$$\frac{1}{2}(\vec\nabla\phi)^2 = \frac{1}{2}\int\widetilde{dp}\int\widetilde{dp}'\; i^2\Big(\vec p\cdot\vec p' \;\ldots\qquad - \vec p\cdot\vec p' \;\ldots \qquad + h.c.\Big)$$

$$\frac{m^2}{2}\phi^2 = \frac{1}{2}\int\widetilde{dp}\int\widetilde{dp}'\;\Big(m^2 \;\ldots \qquad + m^2 \;\ldots\Big)$$

$$(2\pi)^3\delta(\vec p+\vec p') \qquad\qquad (2\pi)^3\delta(\vec p-\vec p')$$

do $\int d^3\vec x$

do $\int \frac{d^3\vec p'}{(2\pi)^3}\frac{1}{2\omega'}$

$$\rightarrow H = \frac{1}{2}\int\frac{d^3\vec p}{(2\pi)^3}\frac{1}{2\omega}\frac{1}{2\omega}\Big(\underbrace{(-\omega^2 - \vec p^2 + m^2)}_{=0} a a' e^{-2i\omega t} + \underbrace{(\omega^2 + \vec p^2 + m^2)}_{=2\omega^2} a a'^* + h.c.\Big)$$

$$\Rightarrow H = \int \frac{d^3\vec p}{(2\pi)^3}\frac{1}{2\omega}\cdot\underbrace{\frac{\omega}{2}(a a^* + a^* a)}_{= \omega a^* a}$$

$\rightarrow$ continuous sum over harmonic oscillators

outlook: quantization $\quad a, a^* \rightarrow$ "ladder" operators $\hat a, \hat a^+$

complex scalar field,

2 real fields $\phi_1, \phi_2 \rightarrow$ one complex field ϕ

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi \phi^*$$ manifestly L-inv (ϕ scalar!)

$\rightarrow$ EOM $\quad (\partial_\mu \partial^\mu \phi + m^2 \phi) = 0$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(a_p e^{-ipx} + b_p^* e^{+ipx} \right)$$

different coefficients (2 d.o.f)

Note: $\mathcal{L}$ is invariant under $\phi \to e^{i\alpha} \phi$ global sym α: const

Noether $\quad \Delta\phi = i\alpha \cdot \phi \quad$ (α infinitesimal)

$\quad \Delta\phi^* = -i\alpha \phi^*$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^* \sim i \left((\partial^\mu \phi^*) \phi - (\partial^\mu \phi) \phi^* \right)$$

is conserved, i.e. $\partial_\mu j^\mu = 0$

relativistic generalization of non-relativistic current

$$\vec{j} = \frac{\hbar}{2mi} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right)$$

1.4. General fields

How do we obtain more general (than scalar) relativistic $\mathcal{L}$?

assume $\mathcal{L}(\phi_i, \partial_\mu \phi_i)$ i: "many" fields

(complex scalar field : $i = 2$)

under LT : $\phi_i(x) \to \phi_i'(x') = \underbrace{M(\Lambda)_{ij}}_{\text{lin. transf. of fields } \phi_i} \phi_j(x)$

$x' = \Lambda x$ or $\phi'(x') = M(\Lambda) \cdot \phi(x)$

make 2nd LT $\quad \underbrace{x \to x' = \Lambda x, \quad x' \to x'' = \Lambda' x'}$

$x \to x'' = \Lambda' \Lambda \, x = \Lambda'' x$ (LT form group)

then $\phi''(x'') = M(\Lambda') \phi'(x') = \underbrace{M(\Lambda') \cdot M(\Lambda) \, \phi(x) \overset{!}{=} M(\Lambda'') \, \phi(x)}$

$\Rightarrow$ $M(\Lambda)$ form a representation of Lorentz group

We can classify the fields according to the (irreducible) representation under which they transform under LT

$\to$ study (Poincaré) Lorentz group

scalar field: trivial representation $M(\Lambda) = 1$

This classification is also done for other (non Lorentz) symmetries, i.e. internal symmetries