Exercise 1. ***Getting to Know the Qubit.***

As seen in the lecture, a qubit is an abstract notion implemented by a quantum mechanical two-level system. It can be in any state $\rho \in \mathcal{S}\left(\mathbb{C}^2\right)$, where $\mathcal{S}(\mathscr{H})$ are the positive semidefinite operators of unit trace on $\mathscr{H}$, also called *density operators*.

The state ρ can be represented by its *Bloch sphere representation*, a vector $\vec{a}$ inside the unit ball in $\mathbb{R}^3$. The correspondance is given by

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right) \; ; \tag{1a}$$

$$a_i = \operatorname{tr}\left(\rho\,\sigma_i\right) \; , \tag{1b}$$

where $\{\sigma_i\}$ are the Pauli matrices.

The canonical basis vectors of $\mathbb{C}^2$ are given by $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

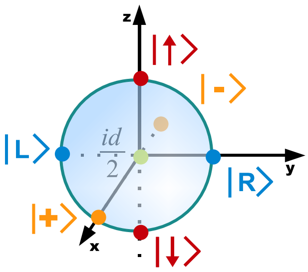
(a) Give the Bloch vectors corresponding to the following pure states, and draw them on the Bloch sphere.

$$|0\rangle \quad ; \quad |1\rangle \quad ; \quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \quad ; \quad |\pm i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i|1\rangle\right) \; .$$

(b) Give the Bloch vectors corresponding to the following states, and draw them on the Bloch sphere:

$$\frac{1}{2}\mathbb{1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \; .$$

Solution. The Bloch vectors are given by the following points on the Bloch ball:



(Figure by Lídia del Rio, from the QIT lecture HS11. The notation is $|\uparrow\rangle = |0\rangle$, $|\downarrow\rangle = |1\rangle$, $|R\rangle = |+\rangle$ and $|L\rangle = |-\rangle$.)

The fully mixed state $\frac{1}{2}\mathbb{1}$ is at the center of the ball. The state $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ is the same as $|0\rangle\langle 0|$, and is at the North pole. The state $\left(\begin{smallmatrix}1/2&1/2\\1/2&1/2\end{smallmatrix}\right)$ is the same as $|+\rangle\langle +|$: indeed,

$$|+\rangle\langle +| = \left(\begin{smallmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{smallmatrix}\right)\left(\begin{smallmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{smallmatrix}\right) = \left(\begin{smallmatrix}1/2&1/2\\1/2&1/2\end{smallmatrix}\right) \; .$$

Rotations on the Bloch sphere correspond to unitaries on $\mathbb{C}^2$ (up to an irrelevant global phase). Recall that the unitaries correspond to a change of orthonormal bases. A unitary U satisfies $U^\dagger U = UU^\dagger = \mathbb{1}$.

(c) Pure states on the qubit (obviously) have the form $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Where are the pure states located in the Bloch sphere representation?

Solution. It is possible to find a unitary U that rotates $|0\rangle$ into $|\psi\rangle$ (since $|\psi\rangle$ can be completed to form an orthonormal basis). Since unitaries correspond to rotations of the Bloch ball, $|\psi\rangle$ has to lie on the surface of the sphere.

The pure states are consequently all on the surface of the Bloch sphere.

(d) Prove relations (1a) and (1b), i.e. that to each vector in the Bloch sphere corresponds a quantum state and vice versa. Argue in particular that the length of the Bloch vector $\vec{a}$ satisfies $|\vec{a}| \leqslant 1$ and that ρ is positive semidefinite.

Hint. The Pauli matrices are $\sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, $\sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. *They satisfy the relation* $\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\sigma_k$ *(*ε_{ijk} *is the signature of the permutation that sends* (123) *to* (ijk) *or zero if two indices are repeated, and we use the Einstein summation convention), as well as* $\operatorname{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$.

Solution. The Pauli matrices are a basis of the 2×2 hermitian traceless matrices. Now, for any ρ, σ density operators, then $\rho - \sigma$ is hermitian and traceless, so if we fix any reference density operator σ, any other density operator ρ can be written as $\rho = \sigma + \Delta$, with Δ being hermitian traceless. One can thus expand Δ in the basis of the Pauli matrices, $\Delta = \sum_i a_i\sigma_i$. In the Bloch representation we choose $\sigma = \frac{1}{2}\mathbb{1}$. This gives expression (1a).

To obtain the components (a_i) of Δ in the basis $\{\sigma_i\}$, we simply project Δ onto the basis elements σ_i using the Hilbert-Schmidt inner product, $\langle \Delta, \sigma_i\rangle = \operatorname{tr}(\Delta\,\sigma_i)$. Since the basis elements are not normalized but have an extra factor of 2 ($\operatorname{tr}\sigma_i\sigma_j = 2\delta_{ij}$), we need to correct for the normalization with a factor $1/2$, $a_i = \frac{1}{2}\operatorname{tr}(\Delta\,\sigma_i)$. Now it is easy to see that $\operatorname{tr}(\rho\,\sigma_i) = \operatorname{tr}(\Delta\,\sigma_i)$ since the σ_i are traceless, and we have (1b).

(This explanation was mathematically more explicit; in this exercise you could just have plugged one expression into the other and checked that you obtained the original thing back.)

It remains to check the domain and ranges of these mappings. Take a Bloch vector $\vec{a}$, i.e. with $|\vec{a}| \leqslant 1$. Consider

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_i a_i\sigma_i\right) = \frac{1}{2}\left[\mathbb{1} + \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix}\right] . \tag{S.1}$$

We have to check that the eigenvalues of ρ are between 0 and 1.

Note that it suffices to verify that the matrix $\left(\begin{smallmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{smallmatrix}\right)$ has eigenvalues between -1 and 1. An eigenvalue λ of this matrix satisfies

$$0 = \begin{vmatrix} a_3 - \lambda & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 - \lambda \end{vmatrix} = -(a_3 - \lambda)(a_3 + \lambda) - (a_1^2 + a_2^2) = \lambda^2 - |\vec{a}|^2 , \tag{S.2}$$

so $\lambda = \pm|\vec{a}| \in [-1, 1]$.

(In particular, the eigenvalues of ρ are $\frac{1}{2} \pm \frac{1}{2}|\vec{a}|$.)

On the other hand, using the fact that $\operatorname{tr}\left(\rho^2\right) \leqslant 1$ (because all eigenvalues λ_i of ρ are between 0 and 1 and $\sum_i \lambda_i = \operatorname{tr}\rho = 1$), we have

$$\begin{aligned} 1 \geqslant \operatorname{tr}\rho^2 = \frac{1}{4}\operatorname{tr}\left[\left(\mathbb{1} + \sum_i a_i\sigma_i\right)\left(\mathbb{1} + \sum_i a_i\sigma_i\right)\right] &= \frac{1}{4}\operatorname{tr}\left[\mathbb{1} + 2\sum_i a_i\sigma_i + \sum_{ij} a_i a_j \sigma_i\sigma_j\right] \\ &= \frac{1}{4}\left(2 + 2\sum_i a_i^2\right) = \frac{1}{2}\left(1 + \sum_i a_i^2\right) , \end{aligned} \tag{S.3}$$

where we used $\operatorname{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$ and $\operatorname{tr}\sigma_i = 0$. We thus have

$$\sum_i a_i^2 \leqslant 1 . \tag{S.4}$$

Exercise 2. ***Measurements on the Qubit.***

A measurement of the qubit along the Z axis will give the result either $+1$ or -1, yielding $+1$ with probability $\frac{1}{2} + \frac{1}{2}a_z$.

(a) Which quantum mechanical observable corresponds to this measurement?

Solution. The measurement projects onto the basis $\{|0\rangle, |1\rangle\}$, so this measurement can be performed by any observable diagonal in this basis, say σ_z, or the spin operator in Z-direction. It can also be the Hamiltonian in case of a Zeeman splitting along the Z axis, for example.

(b) What happens if you measure the qubit, in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, along the Z axis? What is the probability to measure $+1$? and -1? In each case, what is the post-measurement state?

Solution. The state gets projected onto $|0\rangle$ or $|1\rangle$ with respective probabilities $|\langle 0|\psi\rangle|^2$ and $|\langle 1|\psi\rangle|^2$. The probability to measure $|0\rangle$ $(+1)$ is $|\langle 0|\psi\rangle|^2 = \alpha^2$ and the probability to measure $|1\rangle$ (-1) is $|\langle 1|\psi\rangle|^2 = \beta^2$. The post-measurement state is $|0\rangle$ and $|1\rangle$ respectively, as learned in quantum mechanics.

Consider now a quantum mechanical observable A, written in diagonal form as $A = a|a\rangle\langle a| + a'|a'\rangle\langle a'|$, with eigenvalues a and a' and with two orthonormal eigenvectors $|a\rangle$ and $|a'\rangle$.

(c) What happens in the Bloch sphere representation if you measure a qubit with an observable A instead of measuring it along the Z axis?

Solution. The vectors $|a\rangle$ and $|a'\rangle$ form an orthonormal basis, so there must exist a unitary U that performs the change of basis from $\{|0\rangle, |1\rangle\}$ to $\{|a\rangle, |a'\rangle\}$. We know that unitaries are rotations of the Bloch ball, so this means that the measurement is simply performed around another axis rather than the Z axis. In general, (projective) measurements can be performed around any axis projecting the state onto two antipodal points of the sphere.

(d) Let's prepare now the qubit in state $|+\rangle$. What are the outcome probabilities for a measurement along the Z axis?

Solution. Simply calculate $|\langle +|0\rangle|^2 = \frac{1}{2}$ and $|\langle +|1\rangle|^2 = \frac{1}{2}$. Visually, since $|+\rangle$ is on the equator, and since the outcome probabilities are given by the projection onto the Z axis, one can see that the probabilities are $1/2$ for each outcome.

(e) Now let's prepare randomly the qubit in either state $|0\rangle$ or $|1\rangle$ with probability $1/2$. Write the density operator for this system. What are the outcome probabilities for a measurement along the Z axis?

Solution. The density operator is

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} . \tag{S.5}$$

(Notice the difference with point (d), where $\rho = |+\rangle\langle +| = \left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$.)

The outcome probabilities are then $1/2$ for each outcome, since whenever the system is in state $|0\rangle$ it is measured in state $|0\rangle$ with certainty (but this happens with probability $1/2$), and same for when it is prepared in state $|1\rangle$. Visually, this can be seen because the fully mixed state is at the center of the Bloch ball, and the projections on each state $|0\rangle$ and $|1\rangle$ give probabilities of $1/2$ each.

(f) Consider again the two systems given in (d) and (e), but now measure them along the X axis. What happens?

Solution. When we measure $|+\rangle$ along the X axis, i.e. projecting onto states $|\pm\rangle$, then the outcome $|+\rangle$ is observed with certainty. (Indeed, on the Bloch ball, we project $|+\rangle$, which is at position $x = +1$, onto the X axis.)

However, if we project the fully mixed state onto the X axis, we will again obtain the results $|+\rangle$ and $|-\rangle$ with probability $1/2$ each.

This example is meant to emphasize again the difference between the *superposition* between states $|0\rangle$ and $|1\rangle$ (which is again a pure state, $|+\rangle$), and the *statistical mixture* of the two states, which gives the fully mixed state $\frac{1}{2}\mathbb{1}$.