QSIT: Quantum Information Theory.
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Prof. M. Christandl

In quantum state tomography, one often makes statements based on the *likelihood function* $\mathcal{L}(\rho)$. The function is defined for a given dataset obtained during the tomography procedure, and is a function on state space. The likelihood is defined as the probability that from the state ρ, the quantum measurements you performed would give you the actual dataset that you observed,

$$\mathcal{L}(\rho) = \Pr\left[\text{observed data} \,|\, \rho\right] \ . \tag{1}$$

It is often convenient for various applications to consider the logarithm of this function, the loglikelihood $\ell(\rho) = \ln \mathcal{L}(\rho)$.

More generally, the likelihood and loglikelihood are basic quantities for parameter estimation in statistics (not only in quantum mechanics).

One simple tomography procedure, known as *maximum likelihood estimation*, is to take measurements, write down the likelihood function, and report the maximum of this function as being a suitable candidate for an estimation ρ_{MLE} of what the quantum state of our system was. (Technically, one usually prefers for large amount of data to numerically maximize the loglikelihood, since the latter is concave.)

Exercise 1. *State Tomography of a Coin.*

Suppose that you have a biased coin, that gives "heads" with probability p and gives "tails" with probability $1-p$. You would like to estimate this bias, based on a finite number of coin flips.

Suppose that you flip the coin N times, and obtain f_h times heads and $f_t = N - f_h$ times tails.

(a) Write down the likelihood function $\mathcal{L}(p)$ and the loglikelihood function $\ell(p)$ for these measurements, as a function of f_h and f_t. Plot these functions in the interval $p \in [0,1]$.

(b) What would you report as the "true" bias of the coin?

Remark. This problem is identical to measuring only σ_z on an unknown qubit, and estimating the Z-coordinate of its state on the Bloch sphere.

Exercise 2. *Quantum State Tomography on a Qubit.*

Suppose you have a qubit on which you perform tomographic measurements in order to determine its state ρ_{true}. You measure in the Pauli basis, performing n_x measurements of the σ_x observable, n_y of the σ_y observable, and n_z for σ_z. You accumulate statistics by keeping track of the number f_i^+ of "+1"'s measured for the observable σ_i, and the number f_i^- of "−1"'s observed for that observable.

(a) Write down the likelihood function and the loglikelihood function for this procedure, as a function of the f_i^s's. Show that the loglikelihood function is concave.

(b) The *linear inversion* state ρ_{LI} is the one that reproduces the correct probabilities as given by the observed frequences. Calculate this state, which must satisfy

$$\mathrm{tr}\left(\rho_{\mathrm{LI}} \Pi_i^s\right) = \frac{f_i^s}{n_i} \ , \tag{2}$$

where Π_i^s is the POVM effect corresponding to measuring s on the observable σ_i (i.e., it is the projector onto the s eigenspace of σ_i). Is this state always well defined?

Hint. Go to the Bloch sphere picture.

(c) Let ρ_{MLE} be the maximum likelihood estimate. Show that if the linear inversion gives a state ρ_{LI} inside the state space, it coincides with the maximum likelihood estimate ρ_{MLE}.

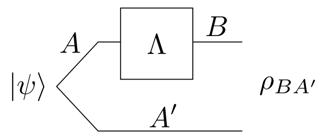
Exercise 3. ***Quantum process tomography***

Imagine that you want to characterize a quantum process (i.e., a TPCPM) $\Lambda_{A\to B}$. For instance, you were given a device by an untrusted party, and you want to verify that it actually does what you were told. The device takes as input a quantum state on system A and outputs states in system B. We will see that we can reduce process tomography to quantum state tomography.

Prepare a maximally entangled state between A and an ancilla system A',

$$|\psi\rangle = \frac{1}{|A|}\sum_i |i\rangle_A|i\rangle_{A'}, \tag{3}$$

and feed the part in A to your device, as displayed schematically in the following figure.



The resulting state is

$$\rho_{BA'} = [\Lambda_{A\to B}\otimes \mathcal{I}_{A'}](|\psi\rangle\langle\psi|_{AA'}). \tag{4}$$

Remark. Equation (4)*, seen as a mapping* $\Lambda_{A\to B}\mapsto \rho_{A'B}$*, can be shown to be an isomorphism mapping the completely positive, trace preserving maps to the density operators. This is known as the* Choi-Jamiolkowski isomorphism*.*

(a) Show that

$$\mathrm{tr}_{A'}\left[\rho_{BA'}(\mathbb{1}_B\otimes |k\rangle\langle \ell|_{A'})\right] = \frac{1}{|A|}\,\Lambda(|\ell\rangle\langle k|). \tag{5}$$

(b) Explain how you would proceed in order to obtain a full characterization of Λ, assuming that you could perform tomography on $\rho_{BA'}$.