Quantum Field Theory I — Problem Set 8

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1. The massive vector field

On this sheet we will develop the QFT for the free massive spin-1 field V_μ.
We start with the Lagrangian for the electromagnetic field and simply add a mass term

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu V^\nu \partial_\mu V_\nu - \tfrac{1}{2}\partial^\mu V^\nu \partial_\nu V_\mu - \tfrac{1}{2}m^2 V^\mu V_\mu. \tag{1}$$

a) Derive the Euler–Lagrange equations of motion for V_μ.

b) By taking a derivative of the equation, show that V_μ is a conserved current.

c) Show that V_μ satisfies the Klein–Gordon equation.

2. Hamiltonian formulation

The Hamiltonian formulation of the massive vector is somewhat tedious due to the presence of constraints.

a) Derive the momenta $\mathit{\Pi}_\mu$ conjugate to the fields V_μ. Considering the space and time components separately, what do you notice?

Your observation is related to constraints. The time component V_0 of the vector field is completely determined by the spatial components and their conjugate momenta (without making reference to time derivatives)

b) Use the equations derived in problem 1 to show that

$$V_0 = -m^{-2}\partial_k \mathit{\Pi}_k, \qquad \dot{V}_0 = \partial_k V_k. \tag{2}$$

c) Substitute this solution for V_0 and $\dot{V}_0$ into the Lagrangian and perform a Legendre transformation to obtain the Hamiltonian. Show that

$$\begin{aligned} H = \int d^3\vec{x}\, \Big(&\tfrac{1}{2}\mathit{\Pi}_k \mathit{\Pi}_k + \tfrac{1}{2}m^{-2}\partial_k \mathit{\Pi}_k \partial_l \mathit{\Pi}_l \\ &+ \tfrac{1}{2}\partial_k V_l \partial_k V_l - \tfrac{1}{2}\partial_l V_k \partial_k V_l + \tfrac{1}{2}m^2 V_k V_k \Big). \end{aligned} \tag{3}$$

d) Derive the Hamiltonian equations of motion for V_k and $\mathit{\Pi}_k$, and compare them to the results of problem 1.

3. Commutators

The unequal time commutators $[V_\mu(x), V_\nu(y)] = \mathit{\Delta}^{\mathrm{V}}_{\mu\nu}(x-y)$ for the massive vector field read

$$\mathit{\Delta}^{\mathrm{V}}_{\mu\nu}(x) = \big(\eta_{\mu\nu} - m^{-2}\partial_\mu\partial_\nu\big)\,\mathit{\Delta}(x), \tag{4}$$

where $\mathit{\Delta}(x)$ is the corresponding function for the scalar field.

a) Show that these obey the equations derived in problem 1.

b) Show explicitly that they obey the constraint equations in 2b), i.e.

$$[m^2 V_0(x) + \partial_k \mathit{\Pi}_k(x), V_\nu(y)] = [\dot{V}_0(x) - \partial_k V_k(x), V_\nu(y)] = 0. \tag{5}$$

c) Confirm that the equal time commutators take the canonical form

$$[V_k(\vec{x}), V_l(\vec{y})] = [\mathit{\Pi}_k(\vec{x}), \mathit{\Pi}_l(\vec{y})] = 0, \qquad [V_k(\vec{x}), \mathit{\Pi}_l(\vec{y})] = i\delta_{kl}\delta^3(\vec{x}-\vec{y}). \tag{6}$$

$\longrightarrow$

4. Coupled Maxwell and Dirac fields

Quantum electrodynamics describes a coupled system of a Maxwell and an electrically charged Dirac field. The Lagrangian density for this theory is given by

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu}. \tag{7}$$

a) Derive the classical equations of motion.

b) Obtain the stress-energy tensor and show that it is invariant under the gauge transformation

$$A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}\alpha(x), \tag{8}$$

$$\psi'(x) = \exp\left[-ie\alpha(x)\right]\psi(x). \tag{9}$$

c) Show that the total energy and momentum of the system are conserved.