1. The Feynman propagator for a real scalar field

Consider a real scalar field $\phi(x)$.

a) Use the Fourier expansion of $\phi(x)$ to show that

$$\mathit{\Delta}_+(x) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})} \exp\bigl(-ie(\vec{p})t - i\vec{p}\cdot\vec{x}\bigr) \tag{1}$$

with $e(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$.

b) Use Cauchy's residue theorem to show that $\mathit{\Delta}_+(x)$ can be also written as

$$\mathit{\Delta}_+(x) = i\int_{C_+} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 + m^2}\,, \tag{2}$$

where the integration over the contour C_+, which is given in the left figure below, corresponds to the (complex) variable p_0.

The Feynman propagator for the real scalar field is defined as

$$G_\mathrm{F}(x-y) = i\theta(x^0 - y^0)\mathit{\Delta}_+(x-y) + i\theta(y^0 - x^0)\mathit{\Delta}_+(y-x). \tag{3}$$

c) Show that it satisfies the defining relation for a propagator

$$(-\partial^2 + m^2)G_\mathrm{F}(x) = \delta^{d+1}(x). \tag{4}$$

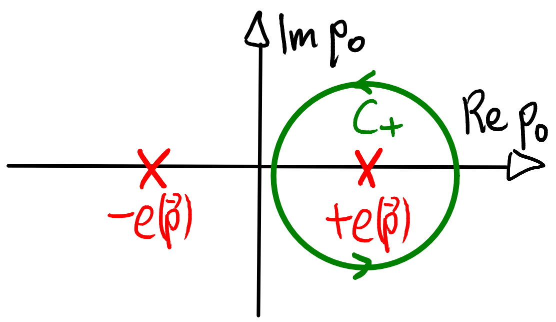
d) Show that

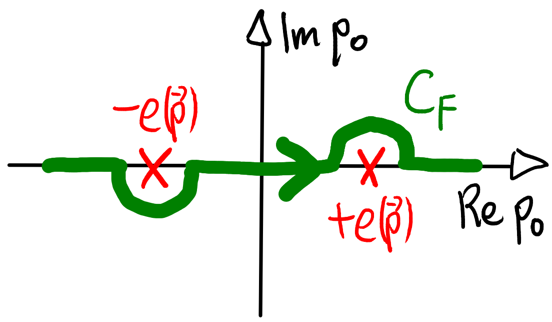
$$G_\mathrm{F}(x) = \int_{C_\mathrm{F}} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 + m^2}\,, \tag{5}$$

with the contour C_F given in the right figure below.

e) Show that the integral in eq.(5) is equivalent to an integral over the real axis

$$G_\mathrm{F}(x) = \int_\mathbb{R} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 + m^2 - i\varepsilon}\,. \tag{6}$$





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2. Conservation of charge with complex scalar fields

Consider a free complex scalar field described by

$$\mathcal{L} = -(\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \tag{7}$$

a) Show that the transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{i\alpha}\phi(x) \tag{8}$$

leaves the Lagrangian density invariant.

b) Find the conserved current associated with this symmetry.

If we now consider two complex scalar fields, the Lagrangian density is given by

$$\mathcal{L} = -(\partial_\mu \phi_a^*)(\partial^\mu \phi^a) - m\phi_a^* \phi^a \qquad a = 1, 2. \tag{9}$$

c) Show that

$$\phi^a(x) \longrightarrow \phi'^a(x) = M^a{}_b \phi^b(x) \tag{10}$$

with $M \in U(2) = \left\{A \in \mathbb{C}^{2\times 2} : A^{-1} = A^\dagger = (A^*)^{\mathrm{T}}\right\}$ is a symmetry transformation.

d) Show that now there are four conserved charges, one given by the generalisation of part b), and the other three given by

$$Q_i = \frac{i}{2} \int d^3\vec{x} \left(\phi_a^* (\sigma^i)^a{}_b \pi^{*b} - \pi_a (\sigma^i)^a{}_b \phi^b\right), \tag{11}$$

where σ^i are the Pauli matrices.

3. Symmetry of the stress-energy tensor

Consider a relativistic scalar field theory specified by some Lagrangian $\mathcal{L}(\phi, \partial\phi)$.

a) Compute the variation of $\mathcal{L}(\phi(x), \partial\phi(x))$ under infinitesimal Lorentz transformations (note: $\omega^{\mu\nu} = -\omega^{\nu\mu}$)

$$x^\mu \longrightarrow x^\mu - \omega^\mu{}_\nu x^\nu. \tag{12}$$

b) Assuming that $\mathcal{L}(x)$ transforms as a scalar field, i.e. just like $\phi(x)$, derive another expression for its variation under Lorentz transformations.

c) Compare the two expressions to show that the two indices of the stress-energy tensor are symmetric

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} = T^{\nu\mu}. \tag{13}$$