3 Scalar Field Quantisation

We can now go ahead and try to quantise the classical scalar field using the canonical procedure described before. We will encounter some infinities, and discuss how to deal with them. Then we shall investigate a few basic objects in QFT.

3.1 Quantisation

Start with the Hamiltonian formulation of the scalar field discussed earlier.

Equal-Time Commutators. Phase space consists of the field $\phi(\vec{x})$ and the conjugate momentum $\pi(\vec{x})$ with Poisson bracket[1]

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \delta^d(\vec{x}-\vec{y}). \tag{3.1}$$

Hence the canonical quantisation implies operators $\phi(\vec{x})$ and $\pi(\vec{x})$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^d(\vec{x}-\vec{y}). \tag{3.2}$$

Note that ϕ and π are now operator-valued fields (rather: distributions).[2]

Field States. Next have to define some states. Straight QM would lead to a state $|\phi_0\rangle$ for every field configuration $\phi_0(\vec{x})$ such that

$$\phi(\vec{x})|\phi_0\rangle = \phi_0(\vec{x})|\phi_0\rangle, \qquad \pi(\vec{x})|\phi_0\rangle = -i\frac{\delta}{\delta\phi_0(\vec{x})}\,|\phi_0\rangle. \tag{3.3}$$

Can be done formally, but not so convenient. Sketch of an eigenstate

$$|0\rangle = \int D\phi\, \exp\left(-\frac{1}{2}\int d^d\vec{x}\, d^d\vec{y}\, \Omega(\vec{x},\vec{y})\, \phi(\vec{x})\, \phi(\vec{y})\right). \tag{3.4}$$

[1] At the moment, the fields are defined on a common time slice t, e.g. $t=0$. Later we discuss unequal times.

[2] The delta-function is a distribution, also the fields should be considered distributions. Distributions are linear maps from test functions to numbers (or operators in this case). In physics, we write them as integrals with a distributional kernel (e.g. the delta-function). Sometimes we also perform illegal operations (e.g. evaluate delta-function $\delta(x)$ at $x=0$).

Momentum Space. The classical field is a bunch of coupled HO's, let us diagonalise them and use creation and annihilation operators.

Go to momentum space, and pick a and $a^\dagger$ appropriately, same transformation as above

$$\begin{aligned}
a(\vec{p}) &= \int d^d\vec{x}\exp\left(i\vec{p}\cdot\vec{x}\right)\big(e(\vec{p})\phi(\vec{x}) + i\pi(\vec{x})\big),\\
\phi(\vec{x}) &= \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a(\vec{p})\exp(-i\vec{p}\cdot\vec{x})\\
&\quad + \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a^\dagger(\vec{p})\exp(+i\vec{p}\cdot\vec{x}),\\
\pi(\vec{x}) &= -\frac{i}{2}\int \frac{d^d\vec{p}}{(2\pi)^d}\, a(\vec{p})\exp(-i\vec{p}\cdot\vec{x})\\
&\quad + \frac{i}{2}\int \frac{d^d\vec{p}}{(2\pi)^d}\, a^\dagger(\vec{p})\exp(+i\vec{p}\cdot\vec{x}).
\end{aligned} \tag{3.5}$$

Commutation relations in momentum space

$$[a(\vec{p}), a^\dagger(\vec{q})] = 2e(\vec{p})\,(2\pi)^d\delta^d(\vec{p}-\vec{q}). \tag{3.6}$$

Substitute fields into Hamiltonian paying attention to ordering

$$\begin{aligned}
H &= \frac{1}{4}\int \frac{d^d\vec{p}}{(2\pi)^d}\big(a^\dagger(\vec{p})a(\vec{p}) + a(\vec{p})a^\dagger(\vec{p})\big)\\
&= \frac{1}{4}\int \frac{d^d\vec{p}}{(2\pi)^d}\big(2a^\dagger(\vec{p})a(\vec{p}) + [a(\vec{p}), a^\dagger(\vec{p})]\big)\\
&= \frac{1}{2}\int \frac{d^d\vec{p}}{(2\pi)^d}\, a^\dagger(\vec{p})a(\vec{p}) + \frac{1}{2}\int d^d\vec{p}\, e(\vec{p})\,\delta^d(\vec{p}-\vec{p}).
\end{aligned} \tag{3.7}$$

Vacuum Energy. Introduce a vacuum state $|0\rangle$ annihilated by all $a(\vec{p})$. Two problems with vacuum energy $H|0\rangle = E_0|0\rangle$

$$E_0 = \frac{1}{2}\int d^d\vec{p}\, e(\vec{p})\,\delta^d(\vec{p}-\vec{p}): \tag{3.8}$$

- delta-function is $\delta^d(\vec{p}-\vec{p}) = \delta^d(0)$ is ill-defined,
- integral $\frac{1}{2}\int d^d\vec{p}\, e(\vec{p})$ diverges.

These are self-made problems:

- We considered an infinite volume. Not reasonable to expect a finite energy. IR problem! The delta-function has units of volume. It measures the volume of the system $\delta^d(\vec{p}-\vec{p}) \sim V \to \infty$. Consider energy density instead!
- Integral $\frac{1}{2}\int_0^P d^d\vec{p}\, e(\vec{p})$ represents vacuum energy density. Infinitely many oscillators per volume element, not reasonable to expect finite energy density. UV problem! Introduce cutoff $|\vec{p}| < P$ to obtain $\frac{1}{2}\int_0^P d^d\vec{p}\, e(\vec{p}) \sim P^{d+1}$.

Finite lattice has similar effect to regularise IR and UV.

To avoid IR infinities in QFT:

- Put the system in a finite box. Possible but inconvenient. Rather work with infinite system and cope with IR divergences. E.g. consider energy density.

To avoid UV infinities in QFT:

- By definition we want a *field* theory, not a discrete model. Physics: Should be able to approximate by discrete model.
- Regularisation: impose momentum cut-off, consider a lattice, other tricks without physical motivation, ...
- Renormalisation: remove terms which would diverge.
- Remove regularisation: obtain finite results.

Here: simply drop E_0.

- There is no meaning to an absolute vacuum energy.
- May add any constant to Hamiltonian (to compensate E_0).
- Makes no observable difference in any physical process.
- Vacuum energy only a philosophical or religious question.
- Later on, infinities lead to interesting effects.

Renormalised Hamiltonian H_{ren}

$$H_{\text{ren}} := H - E_0 = \frac{1}{2}\int \frac{d^d\vec{p}}{(2\pi)^d}\, a^\dagger(\vec{p})a(\vec{p}). \tag{3.9}$$

Nice, vacuum has zero energy.

Normal Ordering. The ordering of variables in classical H plays no role. In quantum theory it does! It is responsible for vacuum energy E_0.

How to map some classical observable O to a quantum operator?

A possible map is normal ordering $\mathrm{N}(O)$:

- Express O in terms of a and $a^* \to a^\dagger$.
- Write all $a^\dagger$'s to the left of all a's.

Here, renormalised Hamiltonian is the normal ordering of H

$$H_{\text{ren}} = \mathrm{N}(H). \tag{3.10}$$

There are other ordering prescriptions for operators:

- Normal ordering depends on the choice of vacuum state. There may be other normal-ordering prescriptions associated to different "vacuum" states.
- There are other useful ordering prescriptions which we will encounter, e.g. time ordering.

3.2 Fock Space

We have:

- a collection of HO's labelled by momentum $\vec{p}$,

- a vacuum state $|0\rangle$ with energy $E = 0$.

Discuss other related states.

Single-Particle States. Can now excite vacuum

$$|\vec{p}\rangle := a^\dagger(\vec{p})|0\rangle. \tag{3.11}$$

Energy eigenstate with energy

$$E = +e(\vec{p}). \tag{3.12}$$

Energy of a particle with momentum $\vec{p}$ and mass m.

Negative-Energy Solutions. Note:

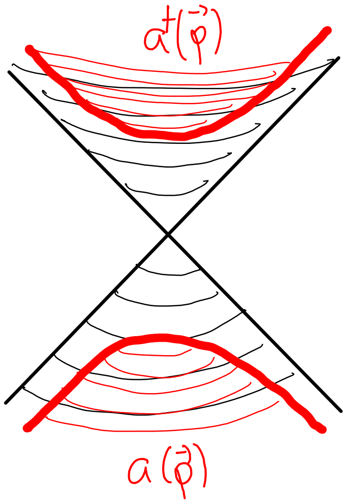
- Energy is positive $E > 0$.
- State $a(\vec{p})|0\rangle$ would have negative energy. Does not exist by construction.

Problem of negative-energy particles solved!

- $a^\dagger(\vec{p})$ creates a particle of momentum $\vec{p}$ a positive energy $+e(\vec{p})$ from the vacuum. $a^\dagger(\vec{p})$ is particle creation operator.
- $a(\vec{p})$ removes a particle of momentum $\vec{p}$ and thus removes the positive energy $+e(\vec{p})$ from the state.

$$\begin{aligned} a(\vec{p})|\vec{q}\rangle &= a(\vec{p})a^\dagger(\vec{q})|0\rangle = [a(\vec{p}), a^\dagger(\vec{q})]|0\rangle \\ &= 2e(\vec{p})\,(2\pi)^d\,\delta^d(\vec{p}-\vec{q})\,|0\rangle. \end{aligned} \tag{3.13}$$

$a(\vec{p})$ is particle annihilation operator.[3]



(3.14)

Interpretation of position space operators not as straight:
$\phi(\vec{x}) = \phi^\dagger(\vec{x})$ as well as $\pi(\vec{x}) = \pi^\dagger(\vec{x})$ create *or* annihilate a particle at position $\vec{x}$.
Result is typically superposition.

[3] $a(\vec{p})$ is not an anti-particle (although this is sometimes claimed). It has negative energy while anti-particles (like particles) have positive energy. For the real scalar, the particle is its own anti-particle.

Normalisation. Let us define proper normalisation for the vacuum

$$\langle 0|0\rangle = 1. \tag{3.15}$$

Normalisation of a single-particle state:

$$\langle \vec{p}|\vec{p}\rangle = 2e(\vec{p})\,(2\pi)^d\,\delta^d(\vec{p}-\vec{p}) = \infty. \tag{3.16}$$

Known problem from QM: Plane-wave states have infinite extent, smeared over all space, unphysical. Recall $\delta^d(\vec{p}-\vec{p})$ represents volume of space.

Consider peaked wave packet state $|f\rangle$ (test function) instead

$$|f\rangle := \int \frac{d^d p\, f(\vec{p})}{(2\pi)^d\, 2e(\vec{p})}\,|\vec{p}\rangle. \tag{3.17}$$

For suitable $f(\vec{p})$ this state has a finite normalisation

$$\langle f|f\rangle = \int \frac{d^d p\, |f(\vec{p})|^2}{(2\pi)^d\, 2e(\vec{p})}\,. \tag{3.18}$$

Multi-Particle States and Fock Space. Now excite more than one harmonic oscillator

$$|\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n\rangle := a^\dagger(\vec{p}_1)\, a^\dagger(\vec{p}_2)\, \cdots\, a^\dagger(\vec{p}_n)\,|0\rangle. \tag{3.19}$$

The energy is given by the sum of particle energies[4]

$$E = \sum_{k=1}^{n} e(\vec{p}_k). \tag{3.20}$$

All particles are freely interchangeable

$$|\ldots, \vec{p}, \vec{q}, \ldots\rangle = |\ldots, \vec{q}, \vec{p}, \ldots\rangle \tag{3.21}$$

because creation operators commute

$$[a^\dagger(\vec{p}), a^\dagger(\vec{q})] = 0. \tag{3.22}$$

Bose statistics for indistinguishable particles: Wave function automatically totally symmetric in QFT.

A generic QFT state (based on the vacuum $|0\rangle$) is a linear combination of k-particle states with k not fixed. This vector space is called Fock space. It is the direct sum

$$\mathbb{V}_{\text{Fock}} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3 \oplus \ldots, \qquad \mathbb{V}_n = (\mathbb{V}_1)^{\otimes_{\mathrm{s}} n} \tag{3.23}$$

of n-particle spaces $\mathbb{V}_n$ where

- $\mathbb{V}_0 = \mathbb{C}$ merely contains the vacuum state $|0\rangle$.

[4] Use $[H, a^\dagger(\vec{p})] = e(\vec{p})a^\dagger(\vec{p})$ to show $H|n\rangle = Ha_1^\dagger \ldots a_n^\dagger|0\rangle = [H, a_1^\dagger]a_2^\dagger \ldots a_n^\dagger|0\rangle + \ldots + a_1^\dagger \ldots a_{n-1}^\dagger[H, a_n^\dagger]|0\rangle = E|n\rangle$.

- $\mathbb{V}_1 = \mathbb{V}_{\text{particle}}$ is the space of single particle states $|\vec{p}\rangle$ with positive energy.
- $\mathbb{V}_n$ is the symmetric tensor product of n copies of $\mathbb{V}_1$.

Consider non-relativistic physics:

- The available energy is bounded from above. Much smaller than particle rest masses $m = e(0)$.
- Relevant part of Fock space with n bounded. For example: $\mathbb{V}_1$, $\mathbb{V}_2$ or $\mathbb{V}_1 \oplus \mathbb{V}_2$.
- Multiple-particle QM is a low-energy limit of QFT. Becomes QFT when number of particles is unbounded.

Conservation Laws. The Hamiltonian H measures the total energy E.

There is also a set of operators to measure total momentum $\vec{P}$

$$\vec{P} := \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, \vec{p}\, a^\dagger(\vec{p})a(\vec{p}). \tag{3.24}$$

with eigenvalue $\vec{P} = \sum_{k=1}^n \vec{p}_k$ on state $|\vec{p}_1, \ldots \vec{p}_n\rangle$. Vacuum carries no momentum. Combine into relativistic vector $P_\mu = (H, \vec{P})$ of operators

$$P_\mu := \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, p_\mu(\vec{p})\, a^\dagger(\vec{p})a(\vec{p}), \qquad p_\mu(\vec{p}) := (e(\vec{p}), \vec{p}). \tag{3.25}$$

Another useful operator is the particle number operator n

$$N := \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a^\dagger(\vec{p})a(\vec{p}). \tag{3.26}$$

It measures the number of particles n in a state $|\vec{p}_1, \ldots \vec{p}_n\rangle$

$$N\mathbb{V}_n = n\mathbb{V}_n. \tag{3.27}$$

The relativistic momentum vector and the number operator are conserved

$$[H, P_\mu] = [H, N] = 0. \tag{3.28}$$

Moreover, they carry no momentum

$$[P_\mu, P_\nu] = [P_\mu, N] = 0. \tag{3.29}$$

In fact, there are many conservation laws. Any two operators made from number density operators commute

$$n(\vec{p}) := a^\dagger(\vec{p})a(\vec{p}), \qquad [n(\vec{p}), n(\vec{q})] = 0. \tag{3.30}$$

Hence such operators are conserved, carry no momentum and no particle number:

$$[H, n(\vec{p})] = [P_\mu, n(\vec{p})] = [N, n(\vec{p})] = 0. \tag{3.31}$$

In a free theory there are infinitely many conservation laws.[5]

[5] Should not say this: Free theory is trivial.

3.3 Complex Scalar Field

Let us discuss a slightly more elaborate case of the scalar field, the complex scalar, where we first encounter anti-particles.

The complex scalar field $\phi(x)$ has the Lagrangian[6] [7]

$$\mathcal{L} = -\partial^\mu \phi^* \, \partial_\mu \phi - m^2 \phi^* \phi = -|\partial\phi|^2 - m^2 |\phi|^2. \tag{3.32}$$

For the conjugate momentum we obtain[8]

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*, \qquad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}. \tag{3.33}$$

In the quantum theory we then impose the canonical commutators

$$[\phi(\vec{x}), \pi(\vec{y})] = [\phi(\vec{x})^\dagger, \pi(\vec{y})^\dagger] = i\delta(\vec{x} - \vec{y}). \tag{3.34}$$

The equation of motion associated to the above Lagrangian is the very same Klein–Gordon equation. However, now complex solutions $\phi(x)$ are allowed. Field operators (with time dependence, see below) now read:

$$\begin{aligned}
\phi(x) &= \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, b(\vec{p}) \exp(-ip{\cdot}x) \\
&\quad + \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a^\dagger(\vec{p}) \exp(+ip{\cdot}x), \\
\phi^\dagger(x) &= \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, a(\vec{p}) \exp(-ip{\cdot}x) \\
&\quad + \int \frac{d^d\vec{p}}{(2\pi)^d \, 2e(\vec{p})} \, b^\dagger(\vec{p}) \exp(+ip{\cdot}x),
\end{aligned} \tag{3.35}$$

Note the strange appearance of a and b. For $a \neq b$ we have $\phi \neq \phi^\dagger$ while $a = b$ implies a real field $\phi = \phi^\dagger$. Non-trivial commutation relations:

$$[a(\vec{p}), a^\dagger(\vec{q})] = [b(\vec{p}), b^\dagger(\vec{q})] = 2e(\vec{p}) \, (2\pi)^d \delta^d(\vec{p} - \vec{q}). \tag{3.36}$$

and quantum Hamiltonian

$$H_{\text{ren}} := \frac{1}{2} \int \frac{d^d\vec{p}}{(2\pi)^d} \big(a^\dagger(\vec{p}) a(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p}) \big). \tag{3.37}$$

The complex scalar carries a charge, $+1$ for ϕ and -1 for $\phi^\dagger$.

---

[6] The prefactor $\frac{1}{2}$ of the real scalar field (ϕ^2) is now absent. It is a convenient symmetry factor. Here the appropriate symmetry factor is 1 because $\phi^* \neq \phi$. More on symmetry factors later.

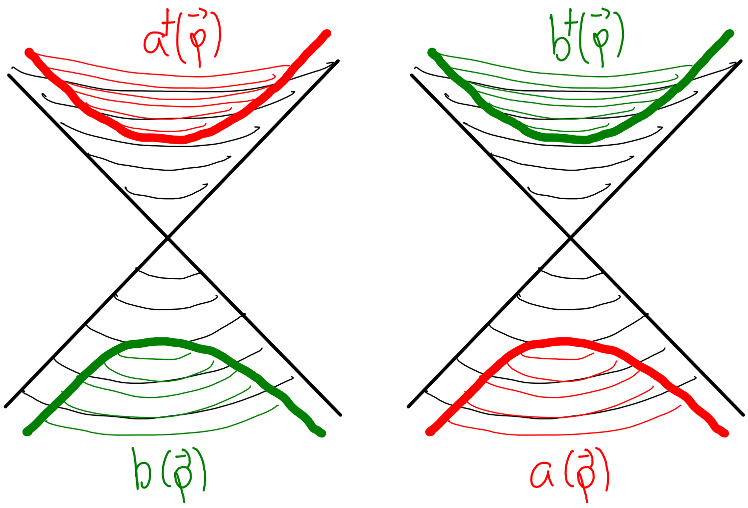
[7] One can also set $\phi(x) = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^*(x) = (\phi_1 - i\phi_2)/\sqrt{2}$ and obtain two independent scalar fields with *equal mass*.

[8] One may just as well define $\pi = \partial\mathcal{L}/\partial\dot{\phi}^* = \dot{\phi}$. It is a matter of convention, and makes no difference if applied consistently.

- The operator $a^\dagger$ creates a particle with charge $+1$.
- The operator b has negative energy, it should remove a particle. The operator carries the same charge as $a^\dagger$, hence the corresponding particle should have charge -1.

Two types of particles: particle and anti-particle. Opposite charges, but equal mass and positive energies. Conclusion:

- $a^\dagger$ creates a particle,
- b annihilates an anti-particle,
- $b^\dagger$ creates an anti-particle,
- a annihilates a particle,
- vacuum annihilated by a's and b's.



(3.38)

3.4 Correlators

Have quantised field. States have an adjustable number of indistinguishable particle with definite momentum. Now what? Consider particle propagation in space and time.

Schrödinger Picture. Steps:

- create a particle at $x^\mu = (t, \vec{x})$,
- let the state evolve for some time $s - t$,
- measure the particle at $y^\mu = (s, \vec{y})$.

Could use $\phi(\vec{x})$ or $\pi(\vec{x})$ to create particle from vacuum.[9] Use ϕ because $\pi = \dot{\phi}$ can be obtained from time derivative.

In Schrödinger picture the states evolve in time

$$i \frac{d}{dt} |\Psi, t\rangle = H |\Psi, t\rangle \tag{3.39}$$

[9]The operators also annihilate particles, but not from the vacuum.

We solve this as[10]

$$|\Psi, s\rangle = \exp\bigl(-iH(s-t)\bigr)|\Psi, t\rangle \tag{3.40}$$

Altogether the correlator reads

$$\begin{aligned}\Delta_+(y,x) &= \langle 0|\phi(\vec{y})\exp\bigl(-i(s-t)H\bigr)\phi(\vec{x})|0\rangle. \\ &= \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, \exp\bigl(-ip{\cdot}(y-x)\bigr).\end{aligned} \tag{3.41}$$

Space and time take different roles. Nevertheless, final answer is perfectly relativistic.

Heisenberg Picture. Heisenberg picture makes relativistic properties more manifest. Translate time-dependence of state to time-dependence of operators

$$F_{\mathrm{H}}(t) := \exp\bigl(+iH(t-t_0)\bigr)F_{\mathrm{S}}(t)\exp\bigl(-iH(t-t_0)\bigr), \tag{3.42}$$

where t_0 is the reference time slice on which quantum states are defined, say $t_0 = 0$. States are therefore time-independent in Heisenberg picture

$$\begin{aligned}F_{\mathrm{S}}(t)|\Psi, t\rangle &= F_{\mathrm{S}}(t)\exp\bigl(-iH(t-t_0)\bigr)|\Psi, t_0\rangle \\ &= \exp\bigl(-iH(t-t_0)\bigr)F_{\mathrm{H}}(t)|\Psi, t_0\rangle.\end{aligned} \tag{3.43}$$

Field ϕ in Heisenberg picture

$$\begin{aligned}\phi(x) = \phi(\vec{x}, t) &= \exp\bigl(+iHt\bigr)\phi(\vec{x})\exp\bigl(-iHt\bigr) \\ &= \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, \bigl(e^{-ip\cdot x}a(\vec{p}) + e^{+ip\cdot x}a^\dagger(\vec{p})\bigr)\end{aligned} \tag{3.44}$$

- has complete spacetime dependence,
- is manifestly relativistic,
- no need to consider $\pi = \dot{\phi}$,
- obeys Klein–Gordon equation $(\partial^2 - m^2)\Delta_+ = 0$,
- same form as solution of Euler–Lagrange equations.

Correlator from Heisenberg picture (vacuum always the same)

$$\Delta_+(y,x) = \langle 0|\phi(y)\phi(x)|0\rangle. \tag{3.45}$$

Same result, but more immediate and relativistic derivation.

Correlator. Let us discuss the correlation function

$$\Delta_+(y,x) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, \exp\bigl(-ip{\cdot}(y-x)\bigr). \tag{3.46}$$

[10] H is time-independent, hence time-translation is governed simply by $\exp(-iH\Delta t)$. Later we shall encounter a more difficult situation.

We know it in momentum space

$$\begin{aligned}\mathit{\Delta}_+(y,x) &= \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\,\mathit{\Delta}_+(p)\exp\bigl(-ip\cdot(y-x)\bigr).\\ \mathit{\Delta}_+(p) &= 2\pi\delta(p^2+m^2)\theta(p_0)\end{aligned} \tag{3.47}$$

How about position space?

- Due to translation symmetry: $\mathit{\Delta}_+(y-x) = \mathit{\Delta}_+(y,x)$.
- Due to Lorentz rotations $\mathit{\Delta}_+(x) = m^{d-1}F(m^2x^2)$.
- Distinguish three regions: future, past, elsewhere.

Klein–Gordon equation for F becomes

$$4rF''(r) + 2(d+1)F'(r) - F(r) = 0 \tag{3.48}$$

This is DG for Bessel functions $J_\alpha(z)$.[11] Two solutions:

$$F_\pm(r) = r^{-(d-1)/4}J_{\pm(d-1)/2}(i\sqrt{r}). \tag{3.49}$$

First, consider future with $x=(t,0)$ where $t=\pm\sqrt{-x^2}$.[12] Substitute in above momentum space expression

$$\begin{aligned}\mathit{\Delta}_+(x) &= \int \frac{d^d\vec{p}}{(2\pi)^d\,2e(\vec{p})}\,\exp\bigl(-ite(\vec{p})\bigr).\\ &= \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\int_0^\infty \frac{dp\,p^{d-1}}{\sqrt{p^2+m^2}}\,\exp\bigl(-it\sqrt{p^2+m^2}\bigr).\\ &= \frac{\mathrm{Vol}(S^{d-1})}{2(2\pi)^d}\int_m^\infty de(e^2-m^2)^{(d-2)/2}\,\exp(-ite)\\ &\sim e^{-imt} \qquad \text{for } t\to\pm\infty.\end{aligned} \tag{3.50}$$

Oscillation with positive frequency m. Same for past. Fixes linear combination of $F_\pm$ (Hankel H_α) [13]

$$\mathit{\Delta}_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}}\,H_{(d-1)/2}\bigl(m\sqrt{-(y-x)^2}\bigr). \tag{3.51}$$

For space-like separation

$$\mathit{\Delta}_+(x) \sim e^{-mr} \qquad \text{for } r=\sqrt{x^2}\to\infty. \tag{3.52}$$

Non-zero, but exponentially decaying (range m). Fixes linear combination of $F_\pm$ (modified Bessel K_α)

$$\mathit{\Delta}_+(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}}\,K_{(d-1)/2}\bigl(m\sqrt{(y-x)^2}\bigr). \tag{3.53}$$

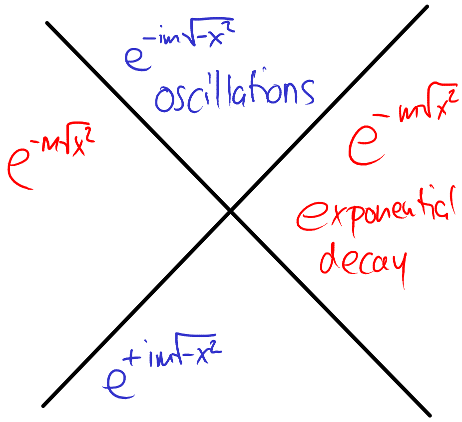
[11] Bessel functions are well-known solutions for spherical waves.

[12] Can go to such a frame with $x=(t,0)$.

[13] The asymptotic behaviour e^{-imt} determines which of the two Hankel functions $H^{(1)}$ and $H^{(2)}$ applies to past and future.

Same as in relativistic QM. Causality?

Before we discuss causality, let us summarise in this figure:



(3.54)

Note that there are also delta-function contributions for light-like separation.

Unequal-Time Commutator. Okay to violate causality as long as not measured.

Correct question: Can one measurement at x influence the other at y? Consider commutator

$$\mathit{\Delta}(y-x) := [\phi(y), \phi(x)]. \tag{3.55}$$

We can relate to above correlators by inserting between vacua

$$\mathit{\Delta}(y-x) = \langle 0|[\phi(y), \phi(x)]|0\rangle = \mathit{\Delta}_+(y-x) - \mathit{\Delta}_+(x-y). \tag{3.56}$$

Observation:

- $\mathit{\Delta}_+$ is symmetric for space-like separations,
- $\phi(x)$ and $\phi(y)$ commute for space-like separations,
- follows also from invariance and equal-time commutator,
- causality preserved.

Note: two contributions cancel for space-like separations:

- Particle created at x and annihilated at y cancels against
- particle created at y and annihilated at x.

However, commutator non-trivial for time-like separations

$$\mathit{\Delta}(y-x) \sim \frac{m^{(d-1)/2}}{(y-x)^{(d-1)/2}}\, J_{(d-1)/2}\big(m\sqrt{-(y-x)^2}\big). \tag{3.57}$$

Time-like separated measurements can influence each other.

Finally we can recover equal-time commutators. For two fields ϕ the commutator follows from symmetry of the integral

$$\mathit{\Delta}(0, \vec{y}-\vec{x}) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})} \big(e^{-i\vec{p}\cdot(\vec{y}-\vec{x})} - e^{i\vec{p}\cdot(\vec{y}-\vec{x})}\big) = 0. \tag{3.58}$$

The commutator between a field ϕ and its conjugate π yields

$$\frac{\partial}{\partial x^0}\mathit{\Delta}(0,\vec{y}-\vec{x}) = \frac{i}{2}\int \frac{d^d\vec{p}}{(2\pi)^d}\left(e^{-i\vec{p}\cdot(\vec{y}-\vec{x})} + e^{i\vec{p}\cdot(\vec{y}-\vec{x})}\right) = i\delta^d(\vec{x}-\vec{y}) \tag{3.59}$$

in agreement with the fundamental commutator $[\phi(\vec{y}),\pi(\vec{x})]$.

3.5 Sources

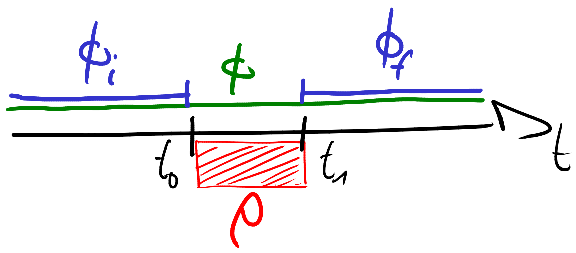
We have a free field, we have discussed correlators of two fields, but there is not much else we can do besides adding interactions (later).

As a first step towards interactions, let us discuss driving the field by an external source $\rho(x)$

$$-\partial^2\phi(x) + m^2\phi(x) = \rho(x). \tag{3.60}$$

The source is non-zero only for some finite interval of time.

Question: Given an initial field $\phi_\mathrm{i}(x)$ obeying the homogeneous Klein–Gordon equation, how to determine the solution $\phi(x)$ of inhomogeneous Klein–Gordon equation such that $\phi(x) = \phi_\mathrm{i}(x)$ for all $t < t_0$ before activation of the source at t_0? In particular, what is the final field $\phi_\mathrm{f}(x)$ which must also obey the homogeneous Klein–Gordon equation.



(3.61)

Due to linearity we make the ansatz[14]

$$\phi(x) = \phi_\mathrm{i}(x) + \mathit{\Delta}\phi(x), \quad \mathit{\Delta}\phi(x) = \int d^{d+1}y\, G_\mathrm{R}(x-y)\,\rho(y) \tag{3.62}$$

where G_R is the retarded propagator or Green('s) function

$$-\partial^2 G_\mathrm{R}(x) + m^2 G_\mathrm{R}(x) = \delta^{d+1}(x), \qquad G_\mathrm{R}(x) = 0 \text{ for } x^0 < 0. \tag{3.63}$$

The first equation is conveniently solved in momentum space

$$G(x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\, e^{-ip\cdot x}\, G(p) \tag{3.64}$$

with Klein–Gordon equation and solution[15]

$$p^2 G(p) + m^2 G_\mathrm{R}(p) = 1, \qquad G(p) = \frac{1}{p^2+m^2}\,. \tag{3.65}$$

[14] Admittedly, this is a perfectly classical problem already encountered in electrodynamics.

[15] As such, this solution is in fact slightly ill-defined, second condition resolves the ambiguities as we shall see.

Let us see how to incorporate the second relation. Write as

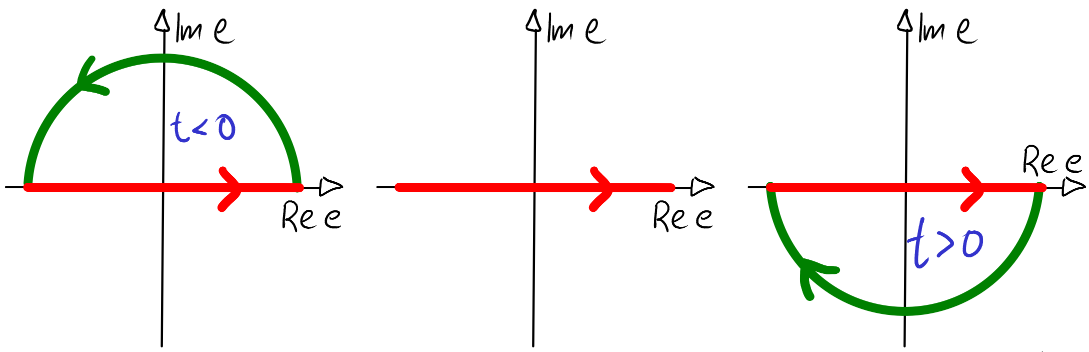
$$G(x) = \int \frac{d^d\vec{p}}{(2\pi)^d}\, e^{-i\vec{p}\cdot\vec{x}} \int \frac{de}{2\pi}\, e^{-iet} \frac{-1}{e^2 - e(\vec{p})^2}$$
$$= \int \frac{d^d\vec{p}}{(2\pi)^d}\, e^{-i\vec{p}\cdot\vec{x}} \int \frac{de}{2\pi}\, \frac{-1}{2e(\vec{p})} \left(\frac{e^{-iet}}{e - e(\vec{p})} - \frac{e^{-iet}}{e + e(\vec{p})} \right). \tag{3.66}$$

Solve Fourier integrals by residue theorem in complex plane. The integral over e runs from $-\infty$ to $+\infty$; close contour! This is done by a semi-circle in the complex plane with very large radius. Contribution must vanish, consider exponent:

$$\exp(-iet) = \exp(-it \operatorname{Re} e) \exp(t \operatorname{Im} e). \tag{3.67}$$

Only second term suppresses contribution:

- For $t > 0$ we need $\operatorname{Im} e < 0$: Close contour in lower half.
- For $t < 0$ we need $\operatorname{Im} e > 0$: Close contour in upper half.

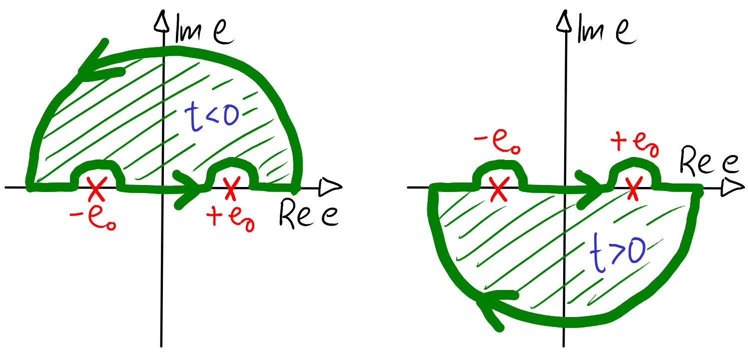


(3.68)

We have two poles at $e = \pm e(\vec{p})$ on the real axis = contour. Need to decide how they contribute to residues.

For retarded propagator, we want $G_{\mathrm{R}}(x) = 0$ for $t < 0$. Achieved by shifting poles slightly into lower half plane[16]

$$G_{\mathrm{R}}(p) = \frac{1}{p^2 + m^2 - ip_0\epsilon}\,. \tag{3.69}$$



(3.70)

No poles in upper half plane, hence $G_{\mathrm{R}}(x)$ for $t < 0$. For $t > 0$, however, both

[16] Alternatively, deform contour to close above points.

poles contribute a residue, we obtain

$$
\begin{aligned}
G_{\mathrm{R}}(x) &= i\theta(t) \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})} \, \big(e^{-ip\cdot x} - e^{+ip\cdot x}\big) \\
&= i\theta(t)\big(\varDelta_+(x) - \varDelta_+(-x)\big) = i\theta(t)\varDelta(x).
\end{aligned} \tag{3.71}
$$

Nice: relation between correlation functions and propagators. Yields position space form for propagator.

Confirm that it satisfies defining relation:[17]

$$
\begin{aligned}
(-\partial^2 + m^2)G_{\mathrm{R}}(x) = i\theta(t)\,(-\partial^2 + m^2)\varDelta(x) &+ \frac{\partial}{\partial t}\big(i\delta(t)\,\varDelta(x)\big) \\
&+ i\delta(t)\,\dot{\varDelta}(x) = \delta^{d+1}(x).
\end{aligned} \tag{3.72}
$$

- First term vanishes because $\varDelta$ is on shell.
- Second term vanishes because $[\phi(\vec{x}), \phi(\vec{y})] = 0$.
- Third term uses $[\phi(\vec{x}), \dot\phi(\vec{y})] = i\delta^d(\vec{x} - \vec{y})$.

We can now determine the contribution to ϕ from the source. Let us focus on the future after the source is switched off[18]

$$
\varDelta\phi(x) = i \int d^{d+1}y\, \varDelta(x-y)\,\rho(y). \tag{3.73}
$$

Transform this to momentum space with

$$
\rho(x) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}}\, e^{-ip\cdot x}\,\rho(p), \quad \rho(p)^* = \rho(-p) \tag{3.74}
$$

yields

$$
\varDelta\phi(x) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})} \big(ie^{-ip\cdot x}\,\rho(p) - ie^{+ip\cdot x}\,\rho^*(p)\big). \tag{3.75}
$$

As we know the homogeneous Klein–Gordon equation is solved by

$$
\phi_{\mathrm{i}}(x) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})} \big(a(\vec{p})e^{-ip\cdot x} + a^\dagger(\vec{p})e^{ip\cdot x}\big). \tag{3.76}
$$

Let this represent the solution in the distant past. Then the solution ϕ_{f} in the distant future is obtained by replacing

$$
a(\vec{p}) \mapsto a(\vec{p}) + i\rho(e(\vec{p}), \vec{p}). \tag{3.77}
$$

We notice that only the Fourier modes of the source ρ on the mass shell can actually drive the field ϕ.

We can now ask how much energy, momentum or particle number the source ρ transfers to the field, e.g.

$$
\varDelta E = \langle 0|H(t > t_1)|0\rangle - \langle 0|H(t < t_0)|0\rangle. \tag{3.78}
$$

[17] Distribute the derivatives in a suitable way between θ and $\varDelta$.

[18] Can therefore set $\theta(y^0 - x^0) = 1$.

The contributions from the quantum modes a, $a^\dagger$ drop out. What remains is manifestly positive for $E = P_0$ and N [19]

$$\begin{aligned} \Delta P_\mu &= \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, p_\mu(\vec{p})\, |\rho(\vec{p})|^2, \\ \Delta N &= \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, |\rho(\vec{p})|^2. \end{aligned} \tag{3.79}$$

[19] ΔN is not manifestly integer for an external field. However, if ρ is quantum, $\rho^\dagger(\vec{p})\rho(\vec{p})$ should again lead to an integer ΔN.