1. **Regge behaviour and hard scattering limit** (intermediate – hard)

We want to have a look at the features of the Veneziano amplitude

\[ A_4 \sim g_s \delta^{26} \left( \sum_i k_i \right) \left( I(s,t) + I(t,u) + I(u,s) \right) \]

where

\[ I(s,t) = \frac{\Gamma(-1 - s\kappa^2)\Gamma(-1 - t\kappa^2)}{\Gamma(-2 - s\kappa^2 - t\kappa^2)} \]

obtained on the last problem sheet by taking two interesting limits.

**a)** The first limit we want to take is \( s \to \infty \) with \( t \) fixed. This is the so called **Regge limit**. Why does this correspond to high energy and small angle scattering? Show that in this limit \( I(s,t) \) reduces to

\[ I(s,t) \sim s^{1+\kappa^2/2} \Gamma(-1 - \kappa^2/2) \]

*Hint:* Use Stirling’s approximation of the \( \Gamma \) function \( \Gamma(1 + x) \approx x^x e^{-x} \sqrt{2\pi x} \).

**b)** The second limit is the hard scattering limit \( s \to \infty \) with \( t/s \) fixed. Argue that this corresponds to high energy and fixed angle scattering. Show that the amplitude reduces to

\[ A_4 \sim \exp(-s \log(\kappa^2 s) - t \log(\kappa^2 t) - u \log(\kappa^2 u)) \]

*Hint:* It might be easier to take the integral expression of the amplitude and attempt a saddle point approximation, but the limit can also be taken straightforwardly.
2. **Low-energy effective action** (easy – intermediate)

In the string frame the low-energy effective action is given by

\[
S = \frac{1}{2\kappa^2} \int d^{26} X \sqrt{-\det G(X)} \ e^{-2\phi} \left( R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \partial_{\mu} \Phi \partial^{\mu} \Phi \right).
\]

Here \( G_{\mu\nu} \) is the metric, \( R \) the associated Ricci scalar, \( H_{\mu\nu\lambda} = 3 \partial_{[\mu} B_{\nu\lambda]} \) is the Kalb-Ramond field strength and \( \Phi \) is a scalar, the dilaton field.

a) Show that the equations of motion of these fields are equivalent to the vanishing of the \( \beta \) functions

\[
\beta_{\mu\nu}(G) = \kappa^2 R_{\mu\nu} + 2\kappa^2 \nabla_{\mu} \nabla_{\nu} \Phi - \frac{\kappa^2}{4} H_{\mu\lambda\sigma} H^{\nu\lambda\sigma},
\]

\[
\beta_{\mu\nu}(B) = -\frac{\kappa^2}{2} \nabla^{\lambda} H_{\lambda\mu\nu} + \kappa^2 \nabla^{\lambda} \Phi H_{\lambda\mu\nu},
\]

\[
\beta(\Phi) = -\frac{\kappa^2}{2} \nabla^{2} \Phi + \kappa^2 \nabla_{\mu} \phi \nabla^{\mu} \Phi - \frac{\kappa^2}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}.
\]

b) The kinetic energy term of the dilaton in the action seems to have the wrong sign. Explain why this is not so.