1. On the importance of quantum gravity (easy)

Let us get some intuition on the order of magnitudes:

a) Consider a gravitational atom, an electron bound to a neutron by the gravitational force. Electromagnetic dipole effects can be neglected. Perform a semiclassical calculation to determine the radius of the orbit of the electron (first Bohr radius). Relate this radius to an appropriate distance in physics.

b) In “natural units”, where $\hbar$, $G$ and $c$ are set to 1, a stellar black hole radiates like a black body at a temperature given by $kT = 1/8\pi M$. Give the temperature in SI units (reinsert $G$, $\hbar$ and $c$) and calculate the temperature of a black hole weighing one solar mass.

2. Relativistic point particle (intermediate)

The action of a relativistic point particle is given by

$$S_{rp} = -\alpha \int_P ds$$

with the relativistic line element

$$ds^2 = -\eta_{\mu\nu} dX^\mu dX^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

and $\alpha$ a (yet to be determined) constant. The path $P$ between two points $X_1^\mu$ and $X_2^\mu$ can be parametrised by a parameter $\tau$. The integral over the line element $ds$ becomes an integral over the parameter

$$S_{rp} = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}}.$$ (1)

a) Parametrise the path by the time coordinate $t$ and take the non-relativistic limit $|\vec{x}| \ll c$ to determine the value of the constant $\alpha$. Characterise the appearing terms.

b) Derive the equations of motion by varying the action in (1). (You may set $c = 1$ from now on.) Hint: Calculate the canonically conjugate momentum $P_\mu$ first.

c) Show that the form of the action is invariant under reparametrisations $\tau' = f(\tau)$. This is what we call manifestly invariant.

d) Consider an electrically charged particle with charge $q$. In the presence of an external gauge field $A_\mu$ there is an additional term in the action governing the interaction between particle and field given by

$$S_{em} = \frac{q}{c} \int d\tau A_\mu(X) \frac{\partial X^\mu}{\partial \tau}.$$ (2)

Find the variation of $A_\mu(X)$ under a variation of the path $\delta X^\mu$. Vary the action $S = S_{rp} + S_{em}$ w.r.t. $X^\mu$ to find the equations of motion for the particle. Hint: Use $P_\mu$ from above to simplify the expression.
3. Polynomial action (intermediate – hard)

There is another way to write the action of a relativistic particle. We introduce an auxiliary field called vierbein (or “einbein” in this case) $e$ along the worldline of the particle and rewrite the action in the form

$$S_{pp} = \int d\tau (e^{-1} \dot{X}^2 - m^2 e).$$

a) Show that $S_{pp}$ is equivalent to $S_{rp}$ above by eliminating the einbein from the action.

b) Derive the equations of motion by varying $S_{pp}$ with respect to $X$ and $e$.

c) Show that $S_{pp}$ is invariant under infinitesimal reparametrisations $\delta \tau = -\epsilon(\tau)$ to linear order in $\epsilon$. First find the correct transformation of $X^\mu$. The einbein transforms like (can you derive it?)

$$\delta e = \partial_\tau (\epsilon(\tau) e).$$

d) Reparametrisation invariance is a gauge invariance. Thus by fixing a gauge we can eliminate one degree of freedom. Assume a gauge in which $e$ is constant. Show that $e$ can be written like

$$e = \frac{\ell}{\tau_2 - \tau_1},$$

where $\ell$ is the invariant length of the worldline for a path starting at $X^\mu(\tau_1)$ and ending at $X^\mu(\tau_2)$. *Hint:* Meditate on the role of the einbein and on how to define $\ell$. 