10 Superstrings

Until now, encountered only bosonic d.o.f. in string theory. Matter in nature is dominantly fermionic. Need to add fermions to string theory.

Several interesting consequences:

- Supersymmetry inevitable.
- Critical dimension reduced from $D = 26$ to $D = 10$.
- Increased stability.
- Closed string tachyon absent. Stable D-branes.
- Several formulations related by dualities.

10.1 Supersymmetry

String theory always includes spin-2 gravitons. Fermions will likely include spin-$\frac{3}{2}$ gravitini $\rightarrow$ supergravity. Spacetime symmetries extended to supersymmetry.

**Super-Poincaré Algebra.** Super-Poincaré algebra is an extension of Poincaré algebra.

Poincaré: Lorentz rotations $M_{\mu\nu}$, translations $P_\mu$.

\[ [M, M] \sim M, \quad [M, P] \sim P, \quad [P, P] = 0. \]

Super-Poincaré: Odd super-translation $Q^I_m \; (a: \text{spinor})$

\[ [M, Q] \sim Q, \quad [Q, P] = 0, \quad \{Q^I_m, Q^J_n\} \sim \delta^{IJ} \gamma^\mu_{\beta\alpha} P_\mu. \]

$N$: rank of supersymmetry $I = 1, \ldots, N$.

$Q$ relates particles of

- of different spin,
- of different statistics,

and attributes similar properties to them. Symmetry between “forces” and “matter”.

More supersymmetry, higher spin particles.

- gauge theory (spin $\leq 1$): $\leq 16 Q$'s.
- gravity theory (spin $\leq 2$): $\leq 32 Q$'s.

**Superspace.** Supersymmetry is symmetry of superspace. Add anticommuting coordinates to spacetime $x^\mu \rightarrow (x^\mu, \theta^I)$. Superfields: expansion in $\theta$ yields various fields

\[ F(x, \theta) = F_0(x) + \theta^I M^I_m(x) + \theta^2 \ldots + \ldots + \theta^{\dim\theta}. \]

Package supermultiplet of particles in a single field.
**Spinors.** Representations of \( Spin(D-1,1) \) (Clifford).

Complex spinors (Dirac) in \((3+1)D\) belong to \( \mathbb{C}^4 \). Can split into chiral spinors (Weyl): \( \mathbb{C}^2 \oplus \mathbb{C}^2 \). Reality condition (Majorana): \( \text{Re}(\mathbb{C}^2 \oplus \mathbb{C}^2) = \mathbb{C}^2 \).

Spinors in higher dimensions:
- spinor dimension times 2 for \( D \to D + 2 \).
- chiral spinors (Weyl) for \( D \) even.
- real spinors (Majorana) for \( D = 0, 1, 2, 3, 4 \) (mod 8).
- real chiral spinors (Majorana–Weyl) for \( D = 2 \) (mod 8).

Maximum dimensions:
- \( D = 10 \): real chiral spinor with 16 components (gauge).
- \( D = 11 \): real spinor with 32 components (gravity bound).

**Super-Yang–Mills Theory.** \( \mathcal{N} = 1 \) supersymmetry in \( D = 10 \) Minkowski space:
- gauge field \( A_\mu \): 8 on-shell d.o.f..
- adjoint real chiral spinor \( \Psi_m \): 8 on-shell d.o.f..

Simple action
\[
S \sim \int d^{10} x \, \text{tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \gamma^\mu \psi^m D_\mu \psi^m \right).
\]

**Supergravity Theories.** Four relevant models:
- \( \mathcal{N} = 1 \) supergravity in 11D: M-Theory.
- \( \mathcal{N} = (1,1) \) supergravity in 10D: Type IIA supergravity.
- \( \mathcal{N} = (2,0) \) supergravity in 10D: Type IIB supergravity.
- \( \mathcal{N} = (1,0) \) supergravity in 10D: Type I supergravity.

Fields always 128+128 d.o.f. (type I: half, SYM only 8+8):

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M-theory has no 2-form and no dilaton: no string theory. Type IIA, IIB and I have 2-form and dilaton: strings?!

**10.2 Green–Schwarz Superstring**

Type II string: Add fermions \( \Theta^{\mu}_I \) to worldsheet. Equal/opposite chirality: IIB/IIA
\textbf{Action.} Supemomentum $\Pi^\mu_{\alpha} = \partial_\alpha X^\mu + \delta^{IJ} \gamma^\mu_{mn} \Theta^m_I \partial_\alpha \Theta^\nu_J$.

\[
S \sim \int d^2 \xi \sqrt{-\text{det} g} g^{\alpha \beta} \Pi^\mu_{\alpha} \Pi^\nu_{\beta} \\
+ \int \left( (\Theta^1 \gamma_\mu d\Theta^1 - \Theta^2 \gamma_\mu d\Theta^2) dX^\mu + \Theta^1 \gamma_\mu d\Theta^1 \Theta^2 \gamma_\mu d\Theta^2 \right).
\]

Action has kappa symmetry (local WS supersymmetry). Only in $D = 10$!

Note: fermions $\Theta$ have first and second class constraints. Non-linear equations of motion. In general difficult to quantise canonically. Conformal gauge does not resolve difficulties.

\textbf{Light-Cone Gauge.} Convenient to apply light-cone gauge. Simplifies drastically: quadratic action, linear e.o.m.

\[
S \sim \int d^2 \xi \left( \partial_L \vec{X} \cdot \partial_R \vec{X} + \frac{1}{2} \Theta_1 \cdot \partial_R \Theta_1 + \frac{1}{2} \Theta_2 \cdot \partial_L \Theta_2 \right)
\]

Bosons $\vec{X}$ with $\partial_L \partial_R \vec{X} = 0$

- Vector of transverse $SO(8)$: $8_v$
- Left and right moving d.o.f.

Fermions $\Theta_1$, $\Theta_2$ with $\partial_R \Theta_1 = 0$ and $\partial_L \Theta_2 = 0$

- Real chiral spinor of transverse $SO(8)$: $8_s$ or $8_c$. Equal/opposite chiralities for IIB/IA: $8_s + 8_s$ or $8_s + 8_c$
- Left and right moving d.o.f. in $\Theta_1$ and $\Theta_2$, respectively.

\textbf{Spectrum.} Vacuum energy and central charge:

- $8$ bosons and $8$ fermions for $L/R$: $a_{L/R} = 8\zeta(1) - 8\zeta(1) = 0$. no shift $a$ for $L_0$
- $c = 10 + 32 \frac{1}{2} = 26$ (fermions count as $\frac{1}{2}$ due to kappa).
- Super-Poincaré anomaly cancels.

Expansion into bosonic modes $\alpha_n$ and fermionic modes $\beta_n$. $n < 0$: creation, $n = 0$: zero mode, $n > 0$: annihilation.

Zero modes and vacuum:

- $\alpha_0$ is c.o.m. momentum: $\vec{q}$.
- $\beta_0$ transforms the vacuum state:

\[
\begin{align*}
\beta \text{ chiral (}8_s\text{)} : & \quad 8_v \leftrightarrow 8_c \quad \text{vacuum} \rightarrow |8_v + 8_s, q\rangle \\
\beta \text{ anti-chiral (}8_c\text{)} : & \quad 8_v \leftrightarrow 8_s \quad \text{vacuum} \rightarrow |8_v + 8_c, q\rangle
\end{align*}
\]

Spectrum at level zero: massless

- Type IIA closed: $(8_v + 8_s) \times (8_v + 8_c)$ (IIA supergravity)

\[
\begin{align*}
8_v \times 8_v + 8_s \times 8_c &= (35_v + 28_v + 1) + (56_v + 8_v), \\
8_v \times 8_s + 8_v \times 8_c &= (56_s + 8_v) + (56_c + 8_s).
\end{align*}
\]
• Type IIB closed: \((8_v + 8_c) \times (8_v + 8_c)\) (IIB supergravity)

\[
8_v \times 8_v + 8_c \times 8_c = (35_v + 28_v + 1) + (35_c + 28_v + 1),
8_v \times 8_s + 8_v \times 8_s = (56_s + 8_c) + (56_s + 8_c).
\]

• Type I closed: \((8_v + 8_c) \times (8_v + 8_c)\) mod \(\mathbb{Z}_2\) (I supergravity)

\[
(35_v + 28_v + 1) + (56_s + 8_c).
\]

• Type I open: \(8_v + 8_c\) (SYM).

10.3 Ramond–Neveu–Schwarz Superstring

There is an alternative formulation for the superstring: RNS. Manifest worldsheet rather than spacetime supersymmetry!

**Action.** Action in conformal gauge:

\[
S \sim \int d^2 \xi \eta_{\mu\nu} \left( \partial_\mu X^\nu \partial_\nu X^\nu + i \Psi_\mu^L \partial_\nu \Psi_\nu^L + i \Psi_\mu^R \partial_\nu \Psi_\nu^R \right).
\]

- action is supersymmetric.
- fermions are worldsheet spinors but spacetime vectors.

Bosons as before. Fermions can be periodic or anti-periodic.

**Ramond Sector.** \(\Psi(\sigma + 2\pi) = \Psi(\sigma)\) periodic.

- Fermion modes \(\beta_n\) as for bosons.
- Vacuum is a real 32-component fermionic spinor.
- \(a = -\frac{1}{2} 8\zeta(1) + \frac{1}{2} 8\zeta(1) = 0\).
- GSO projection: only chiral/anti-chiral states are physical!

**Neveu–Schwarz Sector.** \(\Psi(\sigma + 2\pi) = -\Psi(\sigma)\) anti-periodic.

- Half-integer modes for fermions: \(\beta_{n+1/2}\).
- Vacuum is a bosonic scalar.
- \(a = -\frac{1}{2} 8\zeta(1) - \frac{1}{4} 8\zeta(1) = \frac{1}{2}\).
- GSO projection: physical states require \(\beta^{2n+1}\). No tachyon!

**String Models.** IIB/IIA strings for equal/opposite chiralities in L/R sectors.

**Superconformal Algebra.** (Left) stress-energy tensor and conformal supercurrent:

\[ T_L = \partial_L X \cdot \partial_L X + \frac{i}{2} \Psi_L \cdot \partial_L \Psi_L, \quad J_L = \Psi_L \cdot \partial_L X \]

Superconformal algebra \( L_n, G_r \) (2\( r \) is even/odd for R/NS):

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{8}c m(m^2 - 1)\delta_{m+n}, \\
[L_m, G_r] &= \left(\frac{1}{2}m - r\right)G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}c(r^2 - \frac{1}{4})\delta_{r+s}.
\end{align*}
\]

\( c = D \) (conventional factor \( \frac{3}{2} \) in \( c \) for super-Virasoro).

**Comparison.** GS and RNS approach yield the same results. In light cone gauge: related by \( SO(8) \) triality

Compare features of both approaches:

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</tr>
<tr>
<td>spacetime covariant</td>
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<td>(✓)</td>
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</tbody>
</table>

Third approach exists: Pure spinors (Berkovits). Introduce auxiliary bosonic spinor \( \lambda \) satisfying \( \lambda \gamma^\mu \lambda = 0 \). Shares benefits of GS/RNS; covariant formulation.

### 10.4 Branes

Open superstrings couple to D-branes. Open string spectrum carries D-brane fluctuations.

- massless: \( \mathcal{N} = 1 \) Super-Yang-Mills reduced to \( (d + 1)D \).
- heavy string modes.
- sometimes: scalar tachyon.

**Stable Dp-Branes.** D-branes can be stable or decay. Open string tachyon indicates D-brane instability.

- D-branes in bosonic string theory are unstable.
- Dp-branes for IIB superstring are stable for \( p \) odd.
- Dp-branes for IIA superstring are stable for \( p \) even.
T-duality maps between IIA and IIB.

Stability is related to supersymmetry. Boundary conditions break symmetry

- Lorentz: $SO(9, 1) \rightarrow SO(d, 1) \times SO(9 - d)$.
- 16 supersymmetries preserved for $p$ odd/even in IIB/IA.
- no supersymmetries preserved for $p$ even/odd in IIB/IA.

Supersymmetry removes tachyon; stabilises strings.

Supergravity $p$-Branes. D-branes are non-perturbative objects. Not seen perturbatively due to large mass.

Stable $D_p$-branes have low-energy limit as supergravity solutions.

$p$-brane supported by $(p + 1)$-form, gravity and dilaton.

- IIB/IA have dilaton and two-form (NS-NS sector).
- IIB/IA has forms of even/odd degree (R-R sector); relevant for stable $D_p$-branes.

Features:
- $p$-branes carry $(p + 1)$-form charge. charge prevents $p$-branes from evaporating.
- charge density equals mass density.
- 16/32 supersymmetries preserved. 1/2 BPS condition.
- Non-renormalisation theorem for 1/2 BPS: $p$-branes same at weak/intermediate/strong coupling. BPS $p$-branes describe D$p$-branes exactly.

Type-I Superstring. Consider open strings on D9-branes.

Gravity and gauge anomaly cancellation requires:

- gauge group of dimension 496.
- some special charge lattice property.

Two solutions: $SO(32)$ and $E_8 \times E_8$. Here: $SO(32)$. Breaks 1/2 supersymmetry: Type I.

- Sometimes considered independent type of superstring.
- Or: IIB, 16 D9 branes, space-filling orientifold-plane.

10.5 Heterotic Superstring

Two further superstring theories.

Almost no interaction between left and right movers. Exploit:

- left-movers as for superstring: 10D plus fermions.
- right-movers as for bosonic string: 26D (16 extra).

Heterotic string. 16 supersymmetries.

Anomaly cancellation requires gauge symmetry:

- HET-O: $SO(32)$ or
HET-E: $E_8 \times E_8$.
Gauge group supported by 16 internal d.o.f.
HET-E interesting because $E_8$ contains potential GUT groups:

$$E_5 = SO(10), \quad E_4 = SU(5), \quad E_3 = SU(3) \times SU(2).$$

10.6 Dualities

Dualities relate seemingly different superstring theories.

- T-duality: time vs. space duality on worldsheet.
- S-duality: analog of electro-magnetic duality.

Dualities considered exact because of supersymmetry. Tests.

A Unique Theory. Dualities related various superstrings:

- T-duality: IIA $\leftrightarrow$ IIB; HET-E $\leftrightarrow$ HET-O
- S-duality: HET-O $\leftrightarrow$ Type I; IIB $\leftrightarrow$ IIB

Furthermore IIA and HET-E at strong coupling: 11D supergravity theory (with membrane).

Suspect underlying 11D theory called “M-theory”. Superstring theories as various limits of M-theory.

Mirror Symmetry. Dualities applied to curved string backgrounds: Curved spacetimes with

- inequivalent metrics can have
- equivalent string physics.

E.g.: T-duality between large and small circles. Many examples for Calabi–Yau manifolds.

String/Gauge Duality. Some low-energy effective theories can become exact.

String physics at the location of a brane described exactly by corresponding YM theory.

Example: $N$ coincident D3-branes in IIB string theory. Effective theory: $\mathcal{N} = 4$ Super-Yang–Mills theory in 4D.