7 Conformal Field Theory

So far considered mostly string spectrum:

- equations of motion (local),
- closed/open periodicity conditions (global),
- quantisation.

Quantum mechanics of infinite tower of string modes α_n.

Next will consider local picture on worldsheet: Fields $X(\xi)$. Quantisation → Quantum Field Theory (QFT). Will need for string scattering.

Reparametrisation invariance:

- worldsheet coordinates ξ artificial,
- gauge fixing: conformal gauge,
- worldsheet coordinates ξ meaningful,
- diffeomorphisms → residual conformal symmetry,
- Conformal Field Theory (CFT).

CFT: QFT making use of conformal symmetry.

- do not calculate blindly,
- structure of final results dictated by symmetry,
- conformal symmetry: large amount, exploit!

Let us scrutinise conformal symmetry:

- Central framework in string theory,
- but also useful for many 2D statistical mechanics systems.

7.1 Conformal Transformations

Special coordinate transformation:

- all angles unchanged,
- definition of length can change,

Metric preserved up to scale

$$g'_{\mu'\nu'}(x') = \frac{dx^\mu}{dx'^{\mu'}}\frac{dx^\nu}{dx'^{\nu'}}\, g_{\mu\nu}(x) \overset{!}{=} f(x)\, g_{\mu'\nu'}(x)$$

Action on Coordinates. Generally in D dimensions

- Lorentz rotations $x^\mu \to \Lambda^\mu{}_\nu x^\nu$,
- translations $x^\mu \to x^\mu + a^\mu$,
- scale transformations / dila(ta)tions $x^\mu \to s x^\mu$,
- conformal inversions (discrete) $x^\mu \to x^\mu / x^2$,

- conformal boosts (inversion, translation, inversion).

Conformal group: $SO(D,2)$ (rather: universal cover).

Action on Fields. E.g. a free scalar

$$S \sim \int d^D x\, \tfrac{1}{2}\partial_\mu \Phi(x)\, \partial^\mu \Phi(x).$$

- Manifest invariance under Lorentz rotations & translations

$$\Phi'(x) = \Phi(\Lambda x + a).$$

- Invariance under scaling $x' = sx$ requires

$$\Phi'(x) = s^{(D-2)/2}\, \Phi(sx).$$

- Invariance under inversions

$$\Phi'(x) = (x^2)^{-(D-2)/2}\, \Phi(1/x).$$

Similar (but more complicated) rules for:

- scalar field $\phi(x)$ with different scaling $\phi'(x) = s^{\Delta}\phi(sx)$,
- spinning fields $\rho_\mu, \ldots$,
- derivatives $\partial_\mu \Phi$, $\partial_\mu \partial_\nu \Phi$, $\partial^2 \Phi$, $\ldots$.

2D Conformal Symmetries. QFT's in 2D are rather tractable. CFT's in 2D are especially simple:

- Conformal group splits $SO(2,2) \simeq SL(2,\mathbb{R})_{\mathrm{L}} \times SL(2,\mathbb{R})_{\mathrm{R}}$
- $SL(2,\mathbb{R})_{\mathrm{L/R}}$ act on coordinates as (drop L/R)

$$\xi' = \frac{a\xi + b}{c\xi + d}, \qquad \delta\xi = \beta + \alpha\xi - \gamma\xi^2;$$

  $\beta^{\mathrm{L/R}}$ are two translations, $\alpha^{\mathrm{L/R}}$ are rotations and scaling, $\gamma^{\mathrm{L/R}}$ are two conformal boosts.
- $SL(2,\mathbb{R})_{\mathrm{L/R}}$ extends to infinite-dimensional Virasoro

$$\delta\xi^{\mathrm{L/R}} = \epsilon^{\mathrm{L/R}}(\xi^{\mathrm{L/R}}) = \sum\nolimits_n \epsilon_n^{\mathrm{L/R}}(\xi^{\mathrm{L/R}})^{1-n}.$$

- Boundaries typically distorted by Virasoro. Only subalgebra preserves boundaries, e.g. $SL(2,\mathbb{R})$.

7.2 Conformal Correlators

In a quantum theory interested in

- spectrum of operators (string spectrum),

- probabilities,
- expectation value of operators on states.

In QFT compute (vacuum) expectation values:

- momentum eigenstates: particle scattering, S-matrix

$$\langle \vec{q}_1, \vec{q}_2, \ldots | S | \vec{p}_1, \vec{p}_2, \ldots \rangle = \langle 0 | a(\vec{q}_1) a(\vec{q}_2) \ldots S \ldots a^\dagger(\vec{p}_2) a^\dagger(\vec{p}_1) | 0 \rangle$$

- position eigenstates: time-ordered correlation functions

$$\langle \Phi(x_1) \Phi(x_2) \ldots \rangle = \langle 0 | T[\Phi(x_1) \Phi(x_2) \ldots] | 0 \rangle$$

Correlator of String Coordinates. Can compute a worldsheet correlator using underlying oscillator relations

$$\begin{aligned}\langle 0 | X^\nu(\xi_2) X^\mu(\xi_1) | 0 \rangle = &- \frac{\kappa^2}{2} \eta^{\mu\nu} \log\bigl(\exp(i\xi_2^{\mathrm{L}}) - \exp(i\xi_1^{\mathrm{L}})\bigr) \\ &- \frac{\kappa^2}{2} \eta^{\mu\nu} \log\bigl(\exp(i\xi_2^{\mathrm{R}}) - \exp(i\xi_1^{\mathrm{R}})\bigr) + \ldots\end{aligned}$$

Can reproduce from CFT? Scalar ϕ of dimension Δ:

$$\langle \phi(x_1) \phi(x_2) \rangle = F(x_1, x_2)$$

Correlator should be invariant!

- Translation invariance

$$F(x_1, x_2) = F(x_1 - x_2) =: F(x_{12}).$$

  Just one vector variable.
- Invariance under Lorentz rotations

$$F(x_{12}) = F(x_{12}^2).$$

  Just a scalar variable.
- Scaling invariance

$$\langle \phi(x_1) \phi(x_2) \rangle \overset{!}{=} \langle \phi'(x_1) \phi'(x_2) \rangle = s^{2\Delta} \langle \phi(s x_1) \phi(s x_2) \rangle,$$

  hence $F(x_{12}^2) = s^{2\Delta} F(s^2 x_{12}^2)$ and

$$F(x_{12}^2) = \frac{N}{(x_{12}^2)^\Delta}.$$

  Just a (normalisation) constant N!

Logarithmic Correlator. Our scalar has scaling dimension $\Delta = (D-2)/2 = 0$. Constant correlator $F(x_1, x_2) = N$?! Not quite: Take limit $D = 2 + 2\epsilon$, $N = N_2/\epsilon$

$$F(x_1, x_2) = \frac{N_2}{\epsilon (x_{12}^2)^\epsilon} \to \frac{N_2}{\epsilon} - N_2 \log x_{12}^2 + \ldots .$$

Note: $\Delta = 0$ correlator can be logarithmic. Still not there. Use LC coordinates $x_{12}^2 = -x_{12}^{\mathrm{L}} x_{12}^{\mathrm{R}}$ and identify

$$x^{\mathrm{L}} = \exp(i\xi^{\mathrm{L}}), \qquad x^{\mathrm{R}} = \exp(i\xi^{\mathrm{R}}).$$

Why the identification?

- 2D conformal transformation,
- closed string periodicity $\sigma \equiv \sigma + 2\pi$, but $x^{\mathrm{L/R}}$ unique!
- choose appropriate coordinates for boundaries.

String coordinates are functions of $x^{\mathrm{L/R}}$ except for linear dependence on $\tau = -\frac{i}{2} \log(x^{\mathrm{L}} x^{\mathrm{R}})$. Better choice of field $\partial X^\mu / \partial x^{\mathrm{L/R}}$:

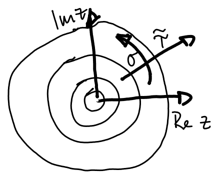
$$\langle 0 | \partial_{\mathrm{L}} X^\nu(x_2) \partial_{\mathrm{L}} X^\mu(x_1) | 0 \rangle = \frac{-\frac{1}{2}\kappa^2 \eta^{\mu\nu}}{\left(x_2^{\mathrm{L}} - x_1^{\mathrm{L}}\right)^2}$$

More manifestly conformal!

Wick Rotation. In this context: Typically perform Wick rotation $\tau = -i\tilde{\tau}$ (now $\tilde{\tau}$ real)

$$\exp(i\xi^{\mathrm{L}}) = \exp(\tilde{\tau} - i\sigma) =: \bar{z}, \quad \exp(i\xi^{\mathrm{R}}) = \exp(\tilde{\tau} + i\sigma) =: z.$$

Cylindrical coordinates for (euclidean) string:



- radius $|z|$ is exponential euclidean time $\tilde{\tau}$,
- σ is angular coordinate (naturally periodic).

Standard treatment: Euclidean CFT

- Worldsheet coordinates z and $\bar{z}$ are complex conjugates.
- Fields are functions $f(z, \bar{z})$ of complex z.
- String coordinates are holomorphic functions

$$X(z, \bar{z}) = X(z) + \bar{X}(\bar{z}).$$

- Conformal transformations are holomorphic.
- Employ powerful functional analysis: residue theorems.

Euclidean WS convenient and conventional. Could as well work on Minkowski worldsheet, nothing lost!

7.3 Local Operators

We understand the basic string coordinate field $X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$, or better $\partial X(z)$ and $\bar{\partial}\bar{X}(\bar{z})$.

Basic objects in a CFT are local operators $\mathcal{O}_i(z, \bar{z})$:

- products of fields X and derivatives $\partial^n \bar{\partial}^{\bar{n}} X$,
- evaluated at the same point $(z, \bar{z})$ on the worldsheet,
- normal ordered $\mathcal{O}_i = :\ldots:$ implicit (no self-correlations),
- for example $\mathcal{O}_1 = :(\partial X)^2:$, $\mathcal{O}_2^{\mu\nu} = :X^\mu \partial X^\nu: - :X^\nu \partial X^\mu:$, ...

Local operators behave

- classically as the sum of constituents,
- quantum-mechanically as independent entities: recall quantum effects in Virasoro charges $(\partial X)^2$!

Main task: classify local operators.

Descendants. All local operators transform under shifts $(\delta z, \delta \bar{z}) = (\epsilon, \bar{\epsilon})$ as

$$\delta \mathcal{O} = \epsilon \partial \mathcal{O} + \bar{\epsilon} \bar{\partial} \mathcal{O}.$$

An operator $\partial^n \bar{\partial}^{\bar{n}} \mathcal{O}$ is called a descendant of $\mathcal{O}$. Shifts are symmetries: No need to consider descendants.

Weights. Most local operators classified by weights $(h, \bar{h})$. Transformation under $(z, \bar{z}) \to (sz, \bar{s}\bar{z})$ or $\delta(z, \bar{z}) = (\epsilon z, \bar{\epsilon} \bar{z})$

$$\mathcal{O}'(z, \bar{z}) = s^h \bar{s}^{\bar{h}} \mathcal{O}(sz, \bar{s}\bar{z}),$$
$$\delta \mathcal{O} = \epsilon(h\mathcal{O} + z\partial \mathcal{O}) + \bar{\epsilon}(\bar{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}).$$

Transformations are scaling and rotation, hence scaling dimension $\Delta = h + \bar{h}$ and spin $S = h - \bar{h}$.

For unitary CFT: Both $h, \bar{h}$ are real and non-negative. E.g. weights: $\partial X \to (1, 0)$, $(\partial X)^2 \to (2, 0)$.

Products of local operators $\mathcal{O} = \mathcal{O}_1 \mathcal{O}_2$:

- total weight is sum of individual weights classically;
- weights usually not additive in quantum theory!

Note: X does not have proper weights, but ∂X does.

Quasi-Primary Operators. A local operator with weights $(h, \bar{h})$ is called quasi-primary if

$$\mathcal{O}'(z, \bar{z}) = \left(\frac{dz'}{dz}\right)^h \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{h}} \mathcal{O}(z', \bar{z}').$$

for all $SL(2, \mathbb{C})$ Möbius transformations

$$z' = \frac{az + b}{cz + d}, \qquad \bar{z}' = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}.$$

For infinitesimal boosts $\delta(z, \bar{z}) = (\epsilon z^2, \bar{\epsilon}\bar{z}^2)$ it must satisfy

$$\delta \mathcal{O} = \epsilon(2hz\mathcal{O} + z^2 \partial \mathcal{O}) + \bar{\epsilon}(2\bar{h}\bar{z}\mathcal{O} + \bar{z}^2 \bar{\partial} \mathcal{O}).$$

Descendants of quasi-primaries are not quasi-primary.
Need to consider only quasi-primary operators.

Primary Operators. An operator is called primary if it satisfies the quasi-primary conditions for all transformations

$$(z,\bar z)\to(z'(z),\bar z'(\bar z))\quad\text{or}\quad(\delta z,\delta\bar z)=(\zeta(z),\bar\zeta(\bar z)).$$

Infinitesimally

$$\delta\mathcal{O}=(h\,\partial\zeta\,\mathcal{O}+\zeta\,\partial\mathcal{O})+(\bar h\,\bar\partial\bar\zeta\,\mathcal{O}+\bar\zeta\,\bar\partial\mathcal{O}).$$

Note: Correlators are only locally invariant. Only a subclass of conformal transformations (e.g. Möbius) leaves correlators globally invariant.

Example. Operator $\mathcal{O}^\mu=\partial X^\mu$ is primary; $(h,\bar h)=(1,0)$.

$$\langle\mathcal{O}_1^\mu\mathcal{O}_2^\nu\rangle=\frac{-\frac{1}{2}\kappa^2\eta^{\mu\nu}}{(z_1-z_2)^2}\,.$$

Invariance under $\delta z=z^{1-n}$:

- exact for $|n|\leq 1$ (Möbius),
- up to polynomials for $|n|>1$ (small w.r.t. $1/(z_1-z_2)^2$).

State-Operator Map. There is a one-to-one map between

- quantum states on a cylinder $\mathbb{R}\times S^1$ and
- local operators (at $z=0$).

Consider the conformal map

$$z=\exp(+i\zeta),\quad\bar z=\exp(-i\bar\zeta),\qquad\zeta,\bar\zeta=\sigma\mp i\tilde\tau.$$

State given by wave function at constant $\tilde\tau=-\operatorname{Im}\zeta$:

- Time evolution is radial evolution in z plane.
- Asymptotic time $\tilde\tau\to-\infty$ corresponds to $z=0$.
- Local operator at $z=0$ to excite asymptotic wave function.
- Unit operator 1 corresponds to vacuum.

7.4 Operator Product Expansion

In a CFT we wish to compute correlation functions

$$\big\langle\mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2)\ldots\mathcal{O}_n(\xi_n)\big\rangle=F_{12\ldots n}.$$

Suppose $\xi_1\approx\xi_2$; then can Taylor expand

$$\mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2)=\sum_{n=0}^{\infty}\frac{1}{n!}(\xi_2-\xi_1)^n\mathcal{O}_1(\xi_1)\partial^n\mathcal{O}_2(\xi_1).$$

Converts local operators at two points into a sum of local operators at a single point. Classical statement is exact.

Quantum OPE. Quantum-mechanically there are additional contributions from operator ordering (normal ordering implicit). Still product of local operators can be written as sum of some local operators

$$\mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2) = \sum\nolimits_i C^i_{12}(\xi_2 - \xi_1)\mathcal{O}_i(\xi_1).$$

More precise formulation with any (non-local) operators "…"

$$\big\langle \mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2)\ldots\big\rangle = \sum\nolimits_i C^i_{12}(\xi_2 - \xi_1)\big\langle \mathcal{O}_i(\xi_1)\ldots\big\rangle.$$

This statement is called Operator Product Expansion (OPE). $C^k_{ij}(\xi_2 - \xi_1)$ are called structure constants & conformal blocks. Sum extends over all local operators (including descendants).

Idea: Every (non-local) operator can be written as an expansion in local operators. Analog: Multipole expansion.

It works exactly in any CFT and is a central tool.

Higher Points. Can formally compute higher-point correlation functions:

$$F_{123\ldots n} = \sum\nolimits_i c^i_{12} F_{i3\ldots n}$$

Apply recursively to reduce to single point.

One-point function is trivial (except for unit operator 1)

$$\langle \mathcal{O}_i\rangle = 0, \quad \langle 1\rangle = 1.$$

Higher-point function reduced to sequence of C^k_{ij}:

- vast simplification,
- need only C^k_{ij} for correlators in CFT,
- hard to compute in practice,
- result superficially depends on OPE sequence (crossing).

Lower Points. Two-point function is OPE onto unity

$$F_{ij} = \langle \mathcal{O}_i\mathcal{O}_j\rangle = \sum\nolimits_k C^k_{ij}\langle \mathcal{O}_k\rangle = C^1_{ij}.$$

Three-point function determines OPE constants

$$F_{ijk} = \langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle = \sum\nolimits_l C^l_{ij}\langle \mathcal{O}_k\mathcal{O}_l\rangle = \sum\nolimits_l F_{kl}C^l_{ij}.$$

Lower-point functions restricted by conformal symmetry:

- Two-point function only for related operators.
- No two-point or three-point conformal invariants. Can map triple of point to any other triple of points.

- Coordinate dependence of two-point function fixed

$$F_{ij} \sim \frac{N_{ij}}{|\xi_i - \xi_j|^{2\Delta_i}}\,.$$

  Numerator N depends on dimension, spin, level of descendant and operator normalisation.
- Coordinate dependence of three-point function fixed

$$F_{ijk} \sim \frac{N_{ijk}}{|\xi_i - \xi_j|^{\Delta_{ij}}|\xi_j - \xi_k|^{\Delta_{jk}}|\xi_k - \xi_i|^{\Delta_{ki}}}$$

  with scaling weights $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$. Numerators N depend on dimension, spin, level of descendant and operator normalisation.
- Three-point functions exist for three different operators.

Normalise operators, then CFT data consists of

- scaling dimensions, spins: spectrum,
- coefficients of three-point function: structure constants.

7.5 Stress-Energy Tensor

The Noether currents for spacetime symmetries are encoded into the conserved stress-energy tensor $T_{\alpha\beta}$

$$T_{\alpha\beta} = -\frac{1}{4\pi\kappa^2}\bigl((\partial_\alpha X)\cdot(\partial_\beta X) - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}(\partial_\gamma X)\cdot(\partial_\delta X)\bigr)$$

Object of central importance for CFT/OPE! Trace is exactly zero: Weyl symmetry. Remaining components T_{LL} and T_{RR} translate to euclidean

$$T = -\frac{1}{\kappa^2}(\partial X)^2, \qquad \bar T = -\frac{1}{\kappa^2}(\bar\partial \bar X)^2.$$

Ignore string physical state condition $T = \bar T = 0$.

Conservation. Current $J(z) = \zeta(z)T(z)$ for $\delta z = \zeta(z)$. Classical conservation $\bar\partial J = 0$ by means of e.o.m.. QFT: Conservation replaced by Ward identity:

$$\bar\partial J(z)\mathcal{O}(w,\bar w) = 2\pi\,\delta^2(z-w,\bar z - \bar w)\,\delta\mathcal{O}(w,\bar w).$$

Current J conserved except at operator locations.

OPE: Integrate z over small ball around w

$$\frac{1}{2\pi}\int_{|z-w|<\epsilon} d^2z\,\ldots$$

Evaluate integration over $\bar z$ ($\int d^2z\bar\partial\ldots = -i\int dz\ldots$)

$$\frac{1}{2\pi i}\int_{|z-w|=\epsilon} dz\,J(z)\mathcal{O}(w,\bar w) = \delta\mathcal{O}(w,\bar w).$$

Similarly for $\bar T$. Consider only holomorphic part.

Stress-Energy OPE. Derive OPE of $\mathcal{O}$ and T.

First consider translation $\delta z = \epsilon$, $\delta\mathcal{O} = \epsilon\partial\mathcal{O}$. Need simple pole to generate residue

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots$$

Further terms with higher poles and polynomials in "…".

Suppose $\mathcal{O}$ has holomorphic weight h. Consider scaling $\delta z = \epsilon z$, $\delta\mathcal{O} = \epsilon(h\mathcal{O} + z\partial\mathcal{O})$. Substitute and require following poles in OPE

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots$$

Next suppose $\mathcal{O}$ is quasi-primary. Consider scaling $\delta z = \epsilon z^2$, $\delta\mathcal{O} = \epsilon(2hz\mathcal{O} + z^2\partial\mathcal{O})$. Substitute and require absence of cubic pole

$$T(z)\mathcal{O}(w,\bar{w}) = \ldots + \frac{0}{(z-w)^3} + \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots$$

Finally suppose $\mathcal{O}$ is primary. Leads to absence of higher poles

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \ldots$$

Note that derivatives shift poles by one order.
Descendants are not (quasi-)primaries.

OPE of stress-energy tensor. Compute explicitly (Wick):

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

Result applies to general CFT's. Virasoro algebra!

- T is a local operator,
- T has holomorphic weight $h = 2$ (classical),
- T is quasi-primary,
- T is not primary (unless $c = 0$),
- quartic pole carries central charge $c = D$.

Conformal transformations for T almost primary:

$$\begin{aligned}
\delta T &= \delta z\,\partial T + 2\,\partial\delta z\,T + \frac{c}{12}\partial^3\delta z,\\
T'(z) &= \left(\frac{dz'}{dz}\right)^2\left(T(z') + \frac{c}{12}S(z',z)\right),\\
S(z',z) &= \left(\frac{d^3z'}{dz^3}\right)\left(\frac{dz'}{dz}\right)^{-1} - \frac{3}{2}\left(\frac{d^2z'}{dz^2}\right)^2\left(\frac{dz'}{dz}\right)^{-2}
\end{aligned}$$

Additional term S is Schwarzian derivative. Zero for Möbius transformations.