Gaussian transformation - effective action

Ising model

$$s_i = \pm s$$

$$Z = \sum_{\{s_i\}} e^{-\frac{\beta}{2}\sum_{i,j} J_{ij} s_i s_j + \beta \sum_i s_i H_i}$$

Gaussian transformation

ϕ_i continuous field

$$Z = C \int_{-\infty}^{+\infty} \left(\prod_{i'} d\phi_{i'} \right) e^{-\beta S(\phi_i, H_i)} = e^{-\beta F}$$

$$S(\phi_i, H_i) = -\frac{1}{2} \sum_{i,j} (J^{-1})_{ij} (\phi_i - H_i)(\phi_j - H_j) - \frac{1}{\beta} \sum_i \ln[2\cosh(\beta s \phi_i)]$$

saddle point approximation

$$Z \approx C e^{-\beta S(\bar{\phi}_i, H_i)}$$

dominated by extremal exponent

$$0 = \left. \frac{\partial S}{\partial \phi_i} \right|_{\phi_i = \bar{\phi}_i}$$

$$\bar{\phi}_i = H_i - s \sum_j J_{ij} \tanh(\beta s \bar{\phi}_j)$$

Gaussian transformation - effective action

saddle point approximation

$$\bar{\phi}_i = H_i - s\sum_j J_{ij}\tanh(\beta s\bar{\phi}_j) \xrightarrow{H_i=0} \bar{\phi} = Jzs\tanh(\beta s\bar{\phi})$$

uniform

equivalent to mean field approximation

$$m = s\tanh(\beta s\bar{\phi}) \quad \Longleftrightarrow \quad \bar{\phi} = Jzm$$

Correlation function: $\Gamma_{ij} = \langle s_i s_j\rangle - \langle s_i\rangle\langle s_j\rangle \approx -k_B T\dfrac{d^2 S(\bar{\phi}_i, H_i)}{dH_i dH_j}$

$$(\Gamma^{-1})_{ij} = \frac{1}{s^2}\cosh^2(\beta s\bar{\phi})\left\{\delta_{ij} + \frac{\beta s^2 J_{ij}}{\cosh^2(\beta s\bar{\phi})}\right\}$$

$$\left.\begin{aligned} \Gamma_{ij} &= \int \frac{d^3q}{(2\pi)^3}\,\Gamma(\vec{q})e^{i\vec{q}\cdot(\vec{r}_i-\vec{r}_j)} \\ J_{ij} &= \int \frac{d^3q}{(2\pi)^3}\,J(\vec{q})e^{i\vec{q}\cdot(\vec{r}_i-\vec{r}_j)} \end{aligned}\right\} \Rightarrow \boxed{\Gamma(\vec{q}) = \frac{k_B T\Gamma_0}{1+\Gamma_0 J(\vec{q})}} \qquad \Gamma_0 = \beta(s^2 - m^2)$$

Gaussian transformation - Ornstein-Zernike

Correlation function: $$\Gamma_{ij} = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

lattice constant

$$\Gamma(\vec{q}) = \frac{k_B T \Gamma_0}{1 + \Gamma_0 J(\vec{q})}$$

$$J(\vec{q}) = \frac{1}{N} \sum_{i,j} J_{ij} e^{-i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} = -2J \sum_{\alpha=1}^{d} \cos q_\alpha a$$

$$\Gamma_0 = \beta(s^2 - m^2)$$

long-distance behavior → small q-limit

$|\vec{r}_i - \vec{r}_i| \gg a$ $\qquad$ $aq \ll 2\pi$

$$J(\vec{q}) \approx -Jz + Jq^2 a^2$$

$$\Gamma(\vec{q}) \approx \frac{k_B T s^2}{k_B(T - T_c) + J s^2 q^2 a^2 + k_B T m^2 / s^2} \rightarrow \boxed{\Gamma(\vec{q}) = \frac{A}{1 + \xi^2 q^2}}$$

Ornstein-Zernike

$$A = \frac{k_B T \xi^2}{J a^2}$$

$$\xi^2 = \frac{J s^2 a^2}{k_B(T - T_c)}$$

$$\boxed{\Gamma_{\vec{r}} = \int \frac{d^3 q}{(2\pi)^3} \Gamma(\vec{q}) e^{i\vec{q}\cdot\vec{r}} = \frac{k_B T}{4\pi J} \frac{e^{-r/\xi}}{r}}$$