Heisenberg ferromagnet

model: nearest-neighbor ferromagnetic coupling

$$\mathcal{H} = -J\sum_{\langle i,j\rangle}\hat{\vec{S}}_i\cdot\hat{\vec{S}}_j = -J\sum_{\langle i,j\rangle}\left[\hat{S}_i^z\hat{S}_j^z + \frac{1}{2}\left\{\hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+\right\}\right]$$

$$= -\frac{J}{V}\sum_{\vec{q}}\gamma_{\vec{q}}\hat{\vec{S}}_{\vec{q}}\cdot\hat{\vec{S}}_{-\vec{q}} = -\frac{J}{V}\sum_{\vec{q}}\gamma_{\vec{q}}\left[\hat{S}_{\vec{q}}^z\hat{S}_{-\vec{q}}^z + \frac{1}{2}\left\{\hat{S}_{\vec{q}}^+\hat{S}_{-\vec{q}}^- + \hat{S}_{\vec{q}}^-\hat{S}_{-\vec{q}}^+\right\}\right]$$

$$s = \frac{\hbar}{2}$$

$$\left.\begin{aligned}\hat{\vec{S}}_i &= \frac{1}{V}\sum_{\vec{q}}\hat{\vec{S}}_{\vec{q}}e^{i\vec{q}\cdot\vec{r}_i}\\ \gamma_{\vec{q}} &= 2\sum_{\alpha=x,y,\dots}\cos(q_\alpha)\end{aligned}\right\}$$

conserved quantity $\hat{\vec{S}}_{tot} = \sum_i\hat{\vec{S}}_i = \hat{\vec{S}}_{\vec{q}=0}$

mean field T_c $k_BT_c = Jz\hbar^2$

periodic boundary conditions

$$\vec{q} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

$$\left[\hat{S}_i^z, \hat{S}_j^\pm\right] = \pm\hbar\delta_{ij}\hat{S}_i^\pm \qquad \left[\hat{S}_i^+, \hat{S}_j^-\right] = 2\hbar\delta_{ij}\hat{S}_i^z$$

$$\left[\hat{S}_{\vec{q}}^z, \hat{S}_{\vec{q}'}^\pm\right] = \pm\hbar\hat{S}_{\vec{q}+\vec{q}'}^\pm \qquad \left[\hat{S}_{\vec{q}}^+, \hat{S}_{\vec{q}'}^-\right] = 2\hbar\hat{S}_{\vec{q}+\vec{q}'}^z$$

Heisenberg ferromagnet

$$\chi(\vec{q}, t-t') = \frac{i}{\hbar}\Theta(t-t')\langle[\hat{S}^-_{\vec{q}}(t), \hat{S}^+_{-\vec{q}}(t')]\rangle\ .$$

equation of motion

$$\begin{aligned}
i\hbar\frac{\partial}{\partial t}\chi(\vec{q}, t-t') &= -\delta(t-t')\langle[\hat{S}^-_{\vec{q}}, \hat{S}^+_{-\vec{q}}]\rangle + \frac{i}{\hbar}\Theta(t-t')\langle[[\hat{S}^-_{\vec{q}}, \mathcal{H}](t), \hat{S}^+_{-\vec{q}}(t')]\rangle \\
&= 2\delta(t-t')\hbar\langle\hat{S}^z_{\vec{q}=0}\rangle \\
&\quad -\frac{i}{\hbar}\Theta(t-t')\frac{J\hbar}{V}\sum_{\vec{q}'}(\gamma_{\vec{q}'} - \gamma_{\vec{q}+\vec{q}'})\left\{\left\langle\left[\hat{S}^z_{\vec{q}'}(t)\hat{S}^-_{\vec{q}-\vec{q}'}(t), \hat{S}^+_{-\vec{q}}(t')\right]\right\rangle\right. \\
&\quad \left. +\langle\left[\hat{S}^-_{\vec{q}+\vec{q}'}(t)\hat{S}^z_{-\vec{q}'}(t), \hat{S}^+_{-\vec{q}}(t')\right]\rangle\right\}\ .
\end{aligned}$$

Tyablikov decoupling

$$\frac{i}{\hbar}\Theta(t-t')\left\langle\left[\hat{S}^z_{\vec{q}'}(t)\hat{S}^-_{\vec{q}-\vec{q}'}(t), \hat{S}^+_{-\vec{q}}(t')\right]\right\rangle \Rightarrow \begin{cases} \hat{S}^z_{\vec{q}'}(t)\hat{S}^-_{\vec{q}-\vec{q}'}(t) \rightarrow \boxed{\langle\hat{S}^z_0\rangle\hat{S}^-_{\vec{q}}(t)\delta_{0,\vec{q}'}} \\ \hat{S}^-_{\vec{q}+\vec{q}'}(t)\hat{S}^z_{\vec{q}'}(t) \rightarrow \boxed{\langle\hat{S}^z_0\rangle\hat{S}^-_{\vec{q}}(t)\delta_{0,\vec{q}'}} \end{cases}$$

Heisenberg ferromagnet

decoupled equation of motion

$$i\hbar\frac{\partial}{\partial t}\chi(\vec{q},t-t') = 2\delta(t-t')\hbar\langle\hat{S}^z_{\vec{q}=0}\rangle - \frac{J\hbar}{V}\langle\hat{S}^z_{\vec{q}=0}\rangle(\gamma_0-\gamma_{\vec{q}})\chi(\vec{q},t-t')$$

Fourier transformation:

$$\int d\tilde{t}\; e^{i\omega\tilde{t}-\eta\tilde{t}}\left[i\frac{\partial}{\partial\tilde{t}} + \frac{J}{V}\langle\hat{S}^z_{\vec{q}=0}\rangle(\gamma_0-\gamma_{\vec{q}})\right]\chi(\vec{q},\tilde{t}) = \int d\tilde{t}\; e^{i\omega\tilde{t}-\eta\tilde{t}}2\delta(t-t')\langle\hat{S}^z_{\vec{q}=0}\rangle$$

$$\Rightarrow \left\{\omega + i\eta + 2\frac{J}{V}\langle\hat{S}^z_0\rangle(\gamma_0-\gamma_{\vec{q}})\right\}\chi(\vec{q},\omega) = 2\langle\hat{S}^z_0\rangle$$

$$\chi(\vec{q},\omega) = \frac{2\langle\hat{S}^z_0\rangle}{\omega + 2\frac{J}{V}\langle\hat{S}^z_0\rangle(\gamma_0-\gamma_{\vec{q}}) + i\eta}\,.$$

Heisenberg ferromagnet

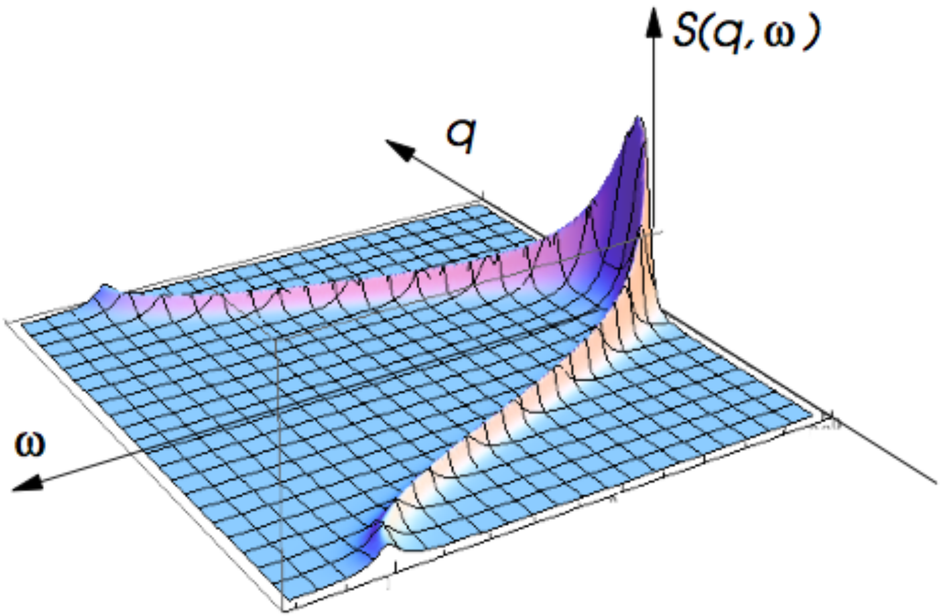
spectrum of excitations $\quad m \neq 0$

$$\chi''(\vec{q},\omega) = 2\pi V m \delta[\omega - 2Jm(\gamma_0 - \gamma_{\vec{q}})]$$

$$\rightarrow \quad \omega_{\vec{q}} = 2Jm(\gamma_0 - \gamma_{\vec{q}}) = 4Jm\sum_{\alpha}(1-\cos q_\alpha)$$

dynamical
structure factor

$$S(\vec{q},\omega) = -\frac{1}{\pi}\frac{\chi''(\vec{q},\omega)}{1-e^{-\beta\hbar\omega}}$$



Heisenberg ferromagnet   instability condition

mean field $\langle \hat{S}_0^z \rangle = -mV$ $\quad m\,(T \to T_{c-}) \to 0$ $\quad k_B T_c^{\mathrm{mf}} \approx 1.5 J\hbar^2$

$$S(\vec{q},\omega) = \int dt\, e^{i\omega t} \frac{1}{\hbar} \langle \hat{S}_{\vec{q}}^-(t) \hat{S}_{-\vec{q}}^+(0) \rangle$$

onsite equal-spin correlation function $\longrightarrow$

$$\frac{1}{V^2} \sum_{\vec{q}} \int d\omega\, S(\vec{q},\omega) = \frac{1}{\hbar} \langle \hat{S}_i^-(0) \hat{S}_i^+(0) \rangle = \frac{1}{\hbar} \left\{ \langle \hat{\vec{S}}_i^2 \rangle - \langle \hat{S}_i^{z2} \rangle - \langle \hat{S}_i^z \rangle \hbar \right\} = \frac{\hbar}{2} + m$$

fluctuation-dissipation theorem

$$\frac{\hbar}{2} + m = -\frac{1}{\pi V^2} \sum_{\vec{q}} \int d\omega \frac{\chi''(\vec{q},\omega)}{1 - e^{-\beta\hbar\omega}}$$

$$= \frac{1}{V} \sum_{\vec{q}} \frac{2m}{1 - e^{-\beta\hbar\omega_{\vec{q}}}}$$

determine *m(T)*

instability $\quad m \to 0$

$$\frac{\hbar}{2} = \frac{k_B T_c}{J\hbar} \frac{1}{V} \sum_{\vec{q}} \frac{1}{\gamma_0 - \gamma_{\vec{q}}}$$

d=3

$$k_B T_c \approx 1.1 J\hbar^2$$

fluctuation renormiert

d= 1,2

$$k_B T_c = 0$$

Hohenberg
Mermin-Wagner

spin wave in ferromagnet - magnon

Heisenberg ferromagnet

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j = -J \sum_{\langle i,j \rangle} \left[\hat{S}_i^z \hat{S}_j^z + \frac{1}{2} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\} \right]$$

ground state: $\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow$ $\quad |\psi^{(0)}\rangle = |\downarrow\downarrow \cdots \downarrow\downarrow\rangle$

$$\langle \hat{S}_i^z \rangle = -\frac{\hbar}{2}$$

$$\mathcal{H}|\psi^{(0)}\rangle = -\frac{J\hbar^2 N}{4}|\psi^{(0)}\rangle = E_0|\psi^{(0)}\rangle$$

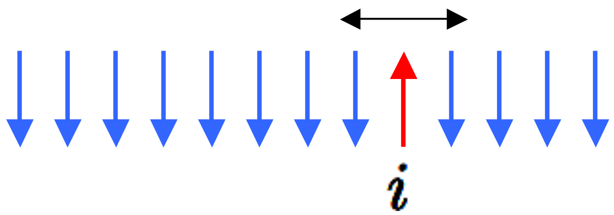
excited state: $\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\underset{i}{\uparrow}\downarrow\downarrow\downarrow\downarrow$ $\quad |\psi_i^{(1)}\rangle$

$$\mathcal{H}_1 = -J \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z \rightarrow \mathcal{H}_1|\psi_i^{(1)}\rangle = E_1|\psi_i^{(1)}\rangle$$

$$E_1 = E_0 + \frac{Jz\hbar^2}{2}$$

spin wave in ferromagnet - magnon

excited state: $\mathcal{H}_1 = -J \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z \rightarrow \mathcal{H}_1 |\psi_i^{(1)}\rangle = E_1 |\psi_i^{(1)}\rangle$



$$|\psi_i^{(1)}\rangle \qquad E_1 = E_0 + \frac{Jz\hbar^2}{2}$$

$$\mathcal{H}_2 = -\frac{J}{2} \sum_{\langle i,j \rangle} \left\{ \hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+ \right\}$$

$$\hat{S}_i^+ |\downarrow\rangle = \hbar |\uparrow\rangle \quad \hat{S}_i^- |\uparrow\rangle = \hbar |\downarrow\rangle$$

"spin hopping"

$$\rightarrow \mathcal{H}_2 |\psi_i^{(1)}\rangle = -\frac{J\hbar^2}{2} \sum_{j \in \Lambda_i} |\psi_j^{(1)}\rangle$$

nearest neighbors of *i*

Fourier transformation $\quad |\phi_{\vec{q}}^{(1)}\rangle = \frac{1}{V} \sum_i |\psi_i^{(1)}\rangle e^{-i\vec{q}\cdot\vec{r}_i} \quad$ "moving particle"

$$\rightarrow \mathcal{H}_2 |\phi_{\vec{q}}^{(1)}\rangle = -\frac{J\hbar^2}{2} \sum_{j \in \Lambda_i} e^{i\vec{q}\cdot(\vec{r}_i - \vec{r}_j)} |\phi_{\vec{q}}^{(1)}\rangle = \frac{J\hbar^2}{2} \gamma_{\vec{q}} |\phi_{\vec{q}}^{(1)}\rangle$$

spin wave in ferromagnet - magnon

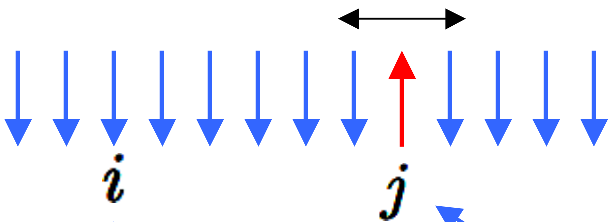
$$\left.\begin{aligned}\mathcal{H}_1|\phi_{\vec{q}}^{(1)}\rangle &= \left(E_0 + \frac{Jz\hbar^2}{2}\right)|\phi_{\vec{q}}^{(1)}\rangle \\ \mathcal{H}_2|\phi_{\vec{q}}^{(1)}\rangle &= \frac{J\hbar^2}{2}\gamma_{\vec{q}}|\phi_{\vec{q}}^{(1)}\rangle\end{aligned}\right\}$$

$$\mathcal{H}|\phi_{\vec{q}}^{(1)}\rangle = \left(E_0 + \frac{J\hbar^2}{2}(\gamma_0 - \gamma_{\vec{q}})\right)|\phi_{\vec{q}}^{(1)}\rangle$$

$$|\phi_{\vec{q}}^{(1)}\rangle = \frac{1}{V}\sum_i |\psi_i^{(1)}\rangle e^{-i\vec{q}\cdot\vec{r}_i}$$

$$\gamma_{\vec{q}} = 2\sum_{\alpha=x,y,z} \cos q_\alpha a$$

$$\Delta E_{\vec{q}} = J\hbar^2 \sum_{\alpha=x,y,z} (1 - \cos q_\alpha a)$$



flip down
measurement

flip up
probe

$$\chi_{ij}(t) = \frac{i}{\hbar}\Theta(t)\langle[\hat{S}_i^-(t), \hat{S}_j^+(0)]\rangle$$