Self-consistent field approximation

partition function from Gaussian transformation

in Ginzburg-Landau formulation (close to T_c) $\bar{\phi} = Jzm$

$$Z = \int \mathcal{D}m(\vec{r}) e^{-\beta F(m;T,H)}$$

functional integral

$$F(m;T,H) = F_0(H,T) + \int d^d r \left\{ \frac{A}{2} m(\vec{r})^2 + \frac{B}{4} m(\vec{r})^4 - H(\vec{r}) m(\vec{r}) + \frac{\kappa}{2} [\vec{\nabla} m(\vec{r})]^2 \right\}$$

expansion around $\langle m(\vec{r}) \rangle = 0$ $T > T_c$ $F_L = 0$

$$F = F_0 + F_L + \frac{1}{2} \int d^d r \, d^d r' \, m(\vec{r}) \left\langle \frac{\delta^2 F}{\delta m(\vec{r}) \delta m(\vec{r}\,')} \right\rangle m(\vec{r}\,')$$

Gaussian approximation

$$\left\langle \frac{\delta^2 F}{\delta m(\vec{r}) \delta m(\vec{r}\,')} \right\rangle = \left\{ A + 3B \langle m(\vec{r})^2 \rangle - \kappa \vec{\nabla}^2 \right\} \delta(\vec{r} - \vec{r}\,')$$

Self-consistent field approximation

$$\left\langle \frac{\delta^2 F}{\delta m(\vec{r})\delta m(\vec{r}\,')} \right\rangle = \left\{ A + 3B\langle m(\vec{r})^2\rangle - \kappa\vec{\nabla}^2 \right\} \delta(\vec{r} - \vec{r}\,')$$

Fourier transformation $m(\vec{r}) = \frac{1}{\sqrt{L^d}} \sum_{\vec{q}} m_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}}$

$$m^*_{\vec{q}} = m_{-\vec{q}}$$

$$m_{\pm\vec{q}} = m'_{\vec{q}} \pm i m''_{\vec{q}}$$

$$F = F_0 + \frac{1}{2}\sum_{\vec{q}} \{A + 3B\langle m^2\rangle + \kappa q^2\}\, m_{\vec{q}} m_{-\vec{q}} = F_0 + \sum_{\vec{q}} G^{-1}(\vec{q}) m_{\vec{q}} m_{-\vec{q}}$$

$$Z' = Z_0 \prod_{\vec{q}} \int dm_{\vec{q}}\, dm_{-\vec{q}}\, \exp\left\{-\beta G^{-1}(\vec{q}) m_{\vec{q}} m_{-\vec{q}}\right\}$$

$$= Z_0 \prod_{\vec{q}} \int dm'_{\vec{q}}\, dm''_{\vec{q}}\, \exp\left\{-\beta G^{-1}(\vec{q}) \left({m'_{\vec{q}}}^2 + {m''_{-\vec{q}}}^2\right)\right\}$$

Self-consistent field approximation

$$F = F_0 + \frac{1}{2}\sum_{\vec{q}}\left\{A + 3B\langle m^2\rangle + \kappa q^2\right\} m_{\vec{q}} m_{-\vec{q}} = F_0 + \sum_{\vec{q}} G^{-1}(\vec{q}) m_{\vec{q}} m_{-\vec{q}}$$

$$Z' = Z_0 \prod_{\vec{q}} \int dm_{\vec{q}}\, dm_{-\vec{q}}\, \exp\left\{-\beta G^{-1}(\vec{q}) m_{\vec{q}} m_{-\vec{q}}\right\}$$

self-consistent field

$$\begin{aligned}\langle m^2\rangle &= \frac{1}{L^d}\int d^d r\, \langle m(\vec{r})^2\rangle = \frac{1}{L^d}\sum_{\vec{q}}\langle m_{\vec{q}} m_{-\vec{q}}\rangle \\ &= \frac{Z_0}{Z' L^d}\sum_{\vec{q}}\prod_{\vec{q}'}\int dm_{\vec{q}'}\, dm_{-\vec{q}'}\, m_{\vec{q}} m_{-\vec{q}} \exp\left\{-\beta G^{-1}(\vec{q}') m_{\vec{q}'} m_{-\vec{q}'}\right\}\end{aligned}$$

$$\Rightarrow \quad \langle m^2\rangle = \frac{k_B T}{L^d}\sum_{\vec{q}} G(\vec{q}) = \frac{1}{L^d}\sum_{\vec{q}} \frac{k_B T}{A + 3B\langle m^2\rangle + \kappa q^2}$$

Self-consistent field approximation - susceptibility

$$\langle m^2 \rangle = \frac{k_B T}{L^d} \sum_{\vec{q}} G(\vec{q}) = \frac{1}{L^d} \sum_{\vec{q}} \frac{k_B T}{A + 3B\langle m^2 \rangle + \kappa q^2}$$

susceptibility

$$\chi(T) = \beta \frac{1}{L^d} \int d^d r \, d^d r' \, \{\langle m(\vec{r}) m(\vec{r}\,') \rangle - \langle m(\vec{r}) \rangle \langle m(\vec{r}\,') \rangle\}$$

$$= \beta \langle m^2_{\vec{q}=0} \rangle = G(\vec{q}=0) = \frac{1}{A + 3B\langle m^2 \rangle} \qquad T > T_c$$

$$\Rightarrow \quad \chi^{-1} = [A + 3B\langle m^2 \rangle] = A + \frac{3Bk_B T}{L^d} \sum_{\vec{q}} \frac{1}{\chi^{-1} + \kappa q^2}$$

Self-consistent field approximation - instabillity

$$\chi^{-1} = [A + 3B\langle m^2\rangle] = A + \frac{3Bk_BT}{L^d}\sum_{\vec{q}}\frac{1}{\chi^{-1}+\kappa q^2}$$

instability at $T \to T_c^*$ $\quad \chi \to \infty \quad \longrightarrow \quad \chi^{-1} \to 0$

$$A = -\frac{3Bk_BT}{L^d}\sum_{\vec{q}}\frac{1}{\kappa q^2}$$

$$\frac{1}{L^d}\sum_{\vec{q}} \to \int \frac{d^dq}{(2\pi)^d} \to \frac{C_d}{(2\pi)^d}\int_0^{\Lambda} dq \frac{q^{d-1}}{\kappa q^2}$$

cutoff: $\Lambda \sim a^{-1}$

$$\begin{aligned} A_c &= Jza^{-d}\left(\frac{T_c^*}{T_c} - 1\right) \\ &= -\frac{Jza^{-d}}{s^2}\frac{C_d k_B T_c^*}{(2\pi)^d}\int_0^{\Lambda} dq\, \frac{q^{d-1}}{\kappa q^2} \end{aligned}$$

$$T_c^* = \frac{T_c}{1 + \frac{C_d z}{(2\pi)^d}\frac{(\Lambda a)^{d-2}}{d-2}}$$

Self-consistent field approximation - critical exponents

$$\chi^{-1} = A + \frac{3Bk_BTC_d}{(2\pi)^d}\int_0^\Lambda dq \frac{q^{d-1}}{\chi^{-1}+\kappa q^2} \longleftrightarrow 0 = A_c + \frac{3Bk_BT_c^*C_d}{(2\pi)^d}\int_0^\Lambda dq \frac{q^{d-1}}{\kappa q^2}$$

temperature dependence of $\chi(T)$ for $T \to T_{c+}^*$

$$\begin{aligned}
\chi^{-1} &= (A-A_c) + \frac{3BC_d}{(2\pi)^d}\int_0^\Lambda dq\, q^{d-1}\left[\frac{k_BT}{\chi^{-1}+\kappa q^2} - \frac{k_BT_c^*}{\kappa q^2}\right] \\
&\approx (A-A_c) - \frac{3BC_dk_BT_c^*}{(2\pi)^d\kappa}\int_0^\Lambda dq \frac{q^{d-3}}{1+\chi\kappa q^2} \qquad \longleftarrow x = q\xi \\
&= (A-A_c) - \frac{3BC_dk_BT_c^*}{(2\pi)^d\kappa}\{\kappa\chi\}^{(2-d)/2}\underbrace{\int_0^{\Lambda\sqrt{\kappa\chi}} dx \frac{x^{d-3}}{1+x^2}}_{\text{cutoff necessary for } d \geq d_c = 4}
\end{aligned}$$

correlation length $\xi^2 = \kappa\chi$

Self-consistent field approximation - critical exponents

$$\boxed{d \geq d_c = 4}$$

$$\chi^{-1} \approx (A - A_c) - \frac{3BC_d k_B T_c^*}{(2\pi)^d \kappa} \{\kappa\chi\}^{(2-d)/2} \frac{\{\Lambda(\kappa\chi)^{1/2}\}^{d-4}}{d-4}$$

$$= \frac{k_B}{a^d s^2} \underbrace{(T - T_c^*)}_{\propto\, \tau} - \frac{C_d z^2}{2(2\pi)^d} \frac{T_c^*}{T_c} \frac{(\Lambda a)^{d-4}}{d-4} \chi^{-1}$$

$$\Rightarrow \quad \chi(T) = \frac{a^d s^2}{k_B(T - T_c^*)} \left\{ 1 + \frac{C_d z^2}{2(2\pi)^d} \frac{T_c^*}{T_c} \frac{(\Lambda a)^{d-4}}{d-4} \right\}^{-1} \propto |\tau|^{-1}$$

$$\chi \propto |\tau|^{-\gamma}$$

$$\xi \propto |\tau|^{-\nu}$$

$\Rightarrow$

$$\xi^2 = \kappa\chi$$

Fisher scaling

$$\gamma = (2 - \eta)\nu$$

$\Rightarrow$

$$\gamma = 1 \qquad \nu = 1/2 \qquad \eta = 0$$

mean field exponents

but

$$T_c^* < T_{\rm cmf}$$

Self-consistent field approximation - critical exponents

$$\boxed{d < d_c = 4}$$

$$\frac{d-2}{2} < 1$$

$$\chi^{-1} = \underbrace{(A - A_c)}_{\propto\, \tau} - \frac{3BC_d k_B T_c^*}{(2\pi)^d \kappa^{d/2}} K_d \chi^{(2-d)/2}$$

regime 1: $\chi^{(2-d)/2}$ dominant T "close" to T_c^*

$$\chi \propto |\tau|^{-\gamma} \qquad \gamma = \frac{2}{d-2}$$

regime 2: χ^{-1} dominant T "far" from T_c^*

$$\chi \propto |\tau|^{-\gamma} \qquad \gamma = 1$$

mean field