# Renormalization group

Analysis of critical phenomena, e.g. at 2nd-order phase transitions

Method: decimation of high-energy degrees of freedom to reach a low-energy effective Hamiltonian without changing the partition function

$$\mathcal{H}(\vec{K},\{s_i\},N) = NK_0 + K_1 \sum_i s_i + K_2 \sum_{\langle i,j\rangle} s_i s_j + \cdots$$

Ising model

$$K_0 = 0\,, \quad K_1 = H/k_BT\,, \quad K_2 = J/K_BT\,, \quad K_{n>2} = 0 \qquad \vec{K} = (K_0, K_1, K_2, \ldots)$$

$$Z(\vec{K},N) = \sum_{\{s_i\}} e^{\mathcal{H}(\vec{K},\{s_i\},N)}$$

separate $\{s_i\} \rightarrow \begin{cases} \{S_b\} & \text{decimate} \\ \{s'\} & \text{keep} \end{cases}$

$$Z(\vec{K},N) = \sum_{\{s'\}} \sum_{\{S_b\}} e^{\mathcal{H}(\vec{K},\{S_b\},\{s'\},N)} = \sum_{\{s'\}} e^{\mathcal{H}(\vec{K}',\{s'\},Nb^{-d})} = Z(\vec{K}',Nb^{-d})$$

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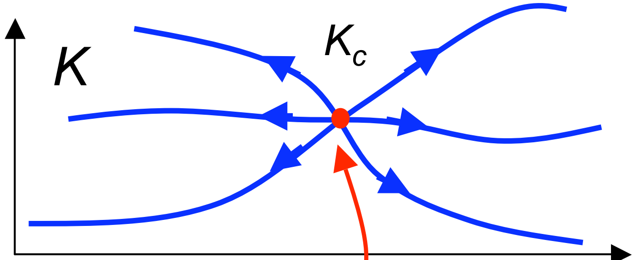
$$Z(\vec{K},N) = \sum_{\{s'\}}\sum_{\{S_b\}} e^{\mathcal{H}(\vec{K},\{S_b\},\{s'\},N)} = \sum_{\{s'\}} e^{\mathcal{H}(\vec{K}',\{s'\},Nb^{-d})} = Z(\vec{K}',Nb^{-d})$$

renormalization group step $\vec{K} \rightarrow R\vec{K} = \vec{K}'$ $\qquad \vec{K}^{(n)} = R^n\vec{K}$

change of length scale $b$ $\qquad \xi \rightarrow \xi' = \xi/b$

number of spins $N \rightarrow N' = N/b^d$

fixed point in flow of $\vec{K}$ $\qquad R\vec{K}_c = \vec{K}_c$




$$\begin{aligned} R\vec{K} &\approx \vec{K}_c + \Lambda(\vec{K}-\vec{K}_c) \\ &= \vec{K}_c + \Lambda\textstyle\sum_i c_i\vec{e}_i \\ &= \vec{K}_c + \textstyle\sum_i c_i b^{y_i}\vec{e}_i \end{aligned}$$

| | | | |
|---|---|---|---|
| $y_i > 0$ | relevant | ➡ | unstable FP |
| $y_i < 0$ | irrelevant | ➡ | stable FP |
| $y_i = 0$ | marginal | | |

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$$R\vec{K} \approx \vec{K}_c + \Lambda(\vec{K} - \vec{K}_c)$$
$$= \vec{K}_c + \Lambda \sum_i c_i \vec{e}_i$$
$$= \vec{K}_c + \sum_i c_i b^{y_i} \vec{e}_i$$

relevant direction $\vec{e}_1$ $\quad y_1 > 0$

$$c_1 = -A\tau \qquad \tau = 1 - T/T_c$$

$$R\tau = \tau' = b^{y_1}\tau$$

correlation length

$$\xi' = \xi / b$$

$$|\tau'|^{-\nu} = \frac{|\tau'|^{-\nu}}{b}$$

$$\Rightarrow \quad \nu = \frac{1}{y_1}$$

specific heat

$$C \propto |\tau|^{-\alpha}$$

$$2 - \alpha = \frac{d}{y_1} = d\nu$$

Josephson scaling

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