Linear Response

small perturbation by external field $\int d^3r \hat{A}(\vec{r}) h(\vec{r},t)$

measured response $\Rightarrow$ $\langle \hat{B}(\vec{r}) \rangle(t) = \int dt' \int d^3r' \, \chi_{BA}(\vec{r}-\vec{r}\,',t-t') h(\vec{r}\,',t')$

Kubo formula / retarded Green's function

$$\chi_{BA}(\vec{r}-\vec{r}\,',t-t') = -\frac{i}{\hbar}\Theta(t-t')\langle[\hat{B}_H(\vec{r},t),\hat{A}_H(\vec{r}\,',t')]\rangle_{\mathcal{H}}$$

causality

$$\langle \hat{C} \rangle_{\mathcal{H}} = \frac{tr\{\hat{C}e^{-\beta\mathcal{H}}\}}{tr\{e^{-\beta\mathcal{H}}\}}$$

thermal average

$$\hat{A}_H(t) = e^{i\mathcal{H}t/\hbar}\hat{A}e^{-i\mathcal{H}t/\hbar}$$

Heisenberg representation

Linear Response

Fourier transform

$$\chi(\vec{q},\omega) = \int d^3\tilde{r} \int_{-\infty}^{+\infty} d\tilde{t}\, \chi(\tilde{\vec{r}},\tilde{t}) e^{i\omega\tilde{t} - i\vec{q}\cdot\tilde{\vec{r}}} \quad \Rightarrow \quad B(\vec{q},\omega) = \chi(\vec{q},\omega) h(\vec{q},\omega)$$

dynamical structure factor

$$\chi(\vec{q},\omega) = \int_0^\infty d\omega'\, S(\vec{q},\omega') \left\{ \frac{1}{\omega - \omega' + i\eta} - \frac{1}{\omega + \omega' + i\eta} \right\}$$

$$S(\vec{q},\omega) = \sum_{n,n'} \frac{e^{-\beta\epsilon_n}}{Z} \underbrace{|\langle n|\hat{B}_{\vec{q}}|n'\rangle|^2 \delta(\hbar\omega - \epsilon_{n'} + \epsilon_n)}_{\text{Fermi Golden rule}}$$

stationary states

$$\mathcal{H}|n\rangle = \epsilon_n |n\rangle$$

and $\hat{A} = \hat{B}^\dagger$

Fermi Golden rule

transition rates between different states of the system

$\eta \rightarrow 0_+$

causality

Linear Response

real- and imaginary part $\chi = \chi' + i\chi''$

$$\chi'(\vec{q},\omega) = \int_0^{\infty} d\omega'\; S(\vec{q},\omega') \left\{ \mathcal{P}\frac{1}{\omega - \omega'} - \mathcal{P}\frac{1}{\omega + \omega'} \right\} ,$$

$$\chi''(\vec{q},\omega) = -\pi \left\{ S(\vec{q},\omega) - S(\vec{q},-\omega) \right\} .$$

Kramers-Kronig relations

$$\chi'(\vec{q},\omega) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega'\; \mathcal{P}\frac{\chi''(\vec{q},\omega')}{\omega - \omega'} ,$$

$$\chi''(\vec{q},\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega'\; \mathcal{P}\frac{\chi'(\vec{q},\omega')}{\omega - \omega'} .$$