Sheet 9

Deadline: 5 December 2011

Exercise 1 [*Dyson series*]:
(i) Show first that

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1\; H_W(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\; H_W(t_1)\; H_W(t_2)$$
$$+ (-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3\, H_W(t_1)\; H_W(t_2)\; H_W(t_3) + \cdots$$

solves the differential equation

$$i\frac{\partial}{\partial t}U(t,t_0) = H_W(t)U(t,t_0) \ .$$

(ii) Then show that one can rewrite $U(t,t_0)$ as

$$\begin{aligned} U(t,t_0) &= 1 + (-i)\int_{t_0}^{t} dt_1 H_W(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, \mathcal{T}\{H_W(t_1)H_W(t_2)\} + \cdots \\ &\equiv \mathcal{T}\left\{\exp\left[-i\int_{t_0}^{t} dt'\, H_W(t')\right]\right\} \ . \end{aligned}$$

Exercise 2 [*Supersymmetry: Wess-Zumino model*]: It is possible to construct field theories with continuous symmetries linking fermions and bosons; such theories are called *supersymmetric*.

(i) The simplest example of a supersymmetric field theory is the *free Wess-Zumino model*, which is the theory of a free complex scalar and a free Weyl fermion. The Lagrangian is

$$\mathcal{L} = \partial_\mu\Phi^*\,\partial^\mu\Phi \,+\, i\chi^\dagger\bar{\sigma}^\mu\partial_\mu\chi \,+\, F^*F \ , \tag{1}$$

where F is an auxiliary complex scalar field. Consider the transformations

$$\begin{aligned} \delta\Phi &= -i\varepsilon^T\sigma^2\chi \\ \delta\chi &= \varepsilon F - \sigma^\mu\partial_\mu\Phi\,\sigma^2\varepsilon^* \\ \delta F &= -i\varepsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi \ , \end{aligned} \tag{2}$$

where ε is a 2-component spinor of *Grassmann* numbers. (Grassmann numbers anti-commute with Grassmann numbers, but commute with regular numbers.) Recalling the identities for $\sigma, \bar{\sigma}$ from Ex. sheet 2, show that the Lagrangian of eq. (1) is invariant (up to a 4-divergence) under this set of transformations.

(ii) We can add an interaction term to the Lagrangian. First we generalise the Wess-Zumino model to n complex scalars and n Weyl spinors as

$$\mathcal{L}_{\text{free}} = \sum_{i=1}^{n} \left[\partial_\mu \Phi_i^* \, \partial^\mu \Phi_i \; + \; i\chi_i^\dagger \bar{\sigma}^\mu \partial_\mu \chi_i \; + \; F_i^* F_i \right] . \tag{3}$$

Next, let the *superpotential* W be a holomorphic function of the Φ_i, $W = W[\Phi_i]$. Show that the Lagrangian

$$\mathcal{L}_{\text{full}} = \mathcal{L}_{\text{free}} + \left(F_i \frac{\partial W[\Phi]}{\partial \Phi_i} + \frac{i}{2} \frac{\partial^2 W[\Phi]}{\partial \Phi_i \partial \Phi_j} \, \chi_i^T \sigma^2 \chi_j \; + \text{h.c.} \right) \tag{4}$$

is invariant under the supersymmetry transformations eq. (2).

Hint: Label the variation of $\mathcal{L}_{\text{full}}$ under SUSY transformations as $\delta\mathcal{L}$. Then

(1) use the Fierz identity

$$(\varepsilon^T \sigma^2 \psi_j)(\psi_k^T \sigma^2 \psi_n) + (\varepsilon^T \sigma^2 \psi_k)(\psi_n^T \sigma^2 \psi_j) + (\varepsilon^T \sigma^2 \psi_n)(\psi_j^T \sigma^2 \psi_k) = 0$$

to show that the term with three fermions in $\delta\mathcal{L}$ vanishes;

(2) show that the terms in $\delta\mathcal{L}$ with one derivative ∂_μ add up to a total divergence;

(3) show then that the terms with F, F^* in $\delta\mathcal{L}$ add to zero.

(iii) For the case $n = 1$, show that a superpotential of the form

$$W[\Phi] = \frac{1}{2} m \, \Phi^2 ,$$

where m is a positive constant, implies that Φ and χ have the same mass.

Hint: Evaluate the equations of motion for F, then for the other fields. What equation do you get for the Weyl spinor?