Differentiable manifold $M \ni p$

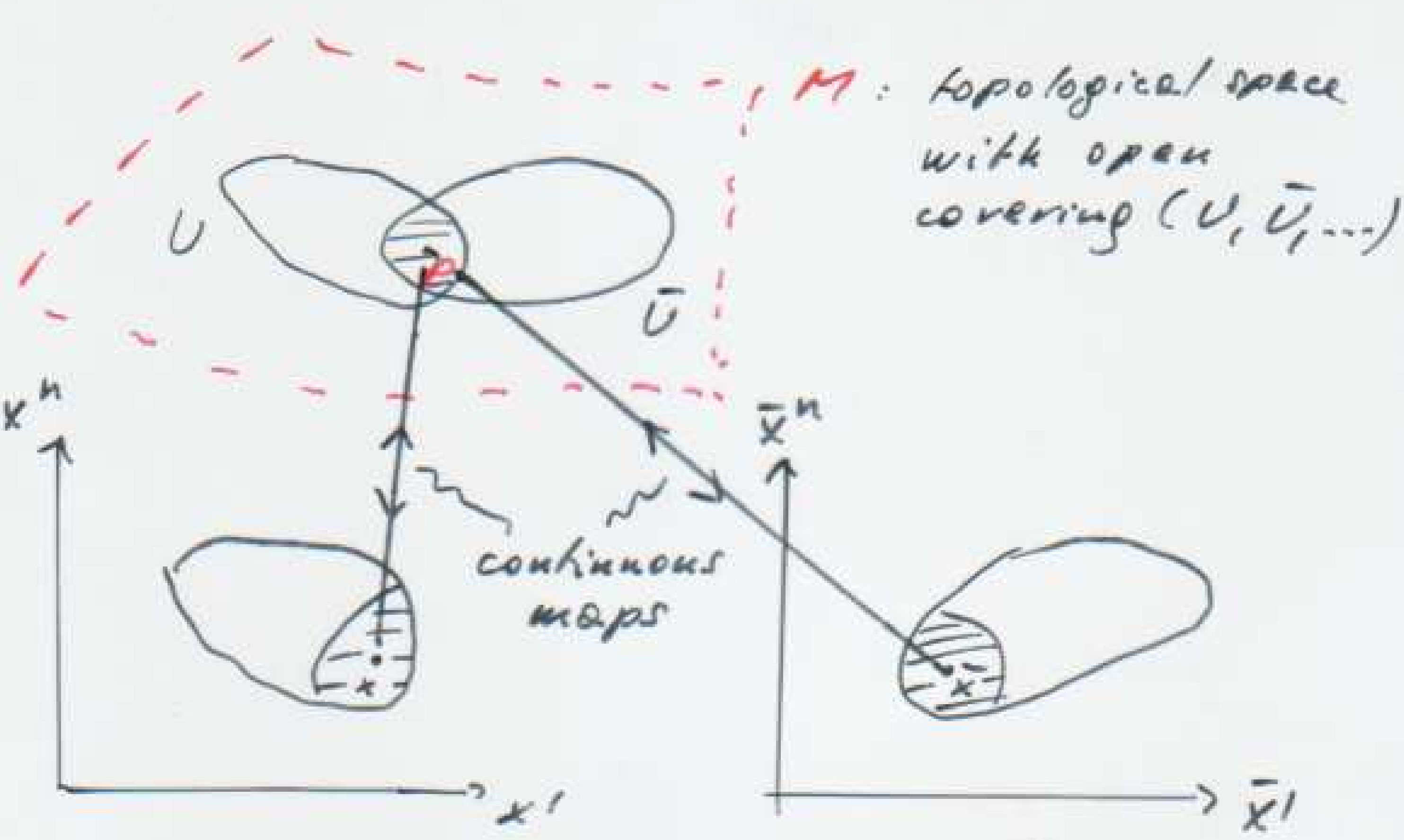


chart $K \subset \mathbb{R}^n$ chart $\bar{K} \subset \mathbb{R}^n$

Change of coordinates : transition functions $\bar{x} \longmapsto x$ are smooth in the (shaded) overlap.

Notions:

- functions $f : M \to \mathbb{R}$
- curves $\gamma : \mathbb{R} \to M$
- maps $\varphi : M \to \bar{M}$

are smooth if they are when represented in some (and hence any) chart(s).

Vectors $X \in T_p$ ← tangent space

$$X : \mathcal{F}_p \to \mathbb{R} \quad , \quad f \mapsto Xf \quad \text{linear}$$

functions defined in a nbhd of p

with "product rule"

$$X(fg) = (Xf)\,g(p) + f(p)(Xg).$$

- Example (general): directional derivative of a curve γ through $p = \gamma(0)$:

$$Xf = \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$$

- In a chart

$$x : U \to K \quad , \quad p' \mapsto x = (x^1, \dots x^n)$$

coordinate functions

we have

$$Xf = X^i f_{,i}(x) \quad , \quad \text{i.e. } X = X^i \frac{\partial}{\partial x^i}$$

$$X^i = X x^i \quad (\in \mathbb{R})$$

- Canonical (or coordinate) basis of T_p:

$$(e_1, \dots e_n) \quad \text{with} \quad e_i = \frac{\partial}{\partial x^i}$$

Covectors $\omega \in T_p^*$ cotangent space

$$\omega : T_p \longrightarrow \mathbb{R}$$
$$X \longmapsto \langle \omega, X \rangle \equiv \omega(X) \quad \text{linear}$$

- Example (general): differential (or gradient) of a function $f \in \mathcal{F}_p$

$$\langle df, X \rangle := Xf$$

- In a chart we have

$$\omega = \omega_i \, dx^i$$
$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\right)$$

- $(e^1, \dots e^n)$ with $e^i = dx^i$

is the dual basis (of T_p^*) to the canonical basis of T_p :

$$\langle dx^i, \frac{\partial}{\partial x^j} \rangle = \delta^i_j$$

- Change of basis

$$\bar{e}_i = \phi_i^{\ k} e_k \qquad\qquad X^i = \phi^i_{\ k} X^k$$

inverse-transposed matrices

$$\bar{e}^i = \phi^i_{\ k} e^k \qquad\qquad \bar{\omega}_i = \phi_i^{\ k} \omega_k$$

- Change of coordinates $x \mapsto \bar{x}$ and hence of canonical bases:

$$\frac{\partial}{\partial \bar{x}^i} = \underbrace{\frac{\partial x^k}{\partial \bar{x}^i}}_{\phi_i^{\ k}} \frac{\partial}{\partial x^k}$$

$$d\bar{x}^i = \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}}_{\phi^i_{\ k}} dx^k$$

Tensors T of type $\binom{p}{q}$ on T_p

$(p, q = 0, 1, 2, \dots)$

e.g. T of type $\binom{1}{2}$

$$T : T_p^* \times T_p \times T_p \to \mathbb{R}$$
$$(\omega, X, Y) \mapsto T(\omega, X, Y)$$

linear in each argument.

- generalize vectors, covectors, numbers:

  type $\binom{0}{1} = T_p^*$ (covectors)

  " $\binom{1}{0} = T_p^{**} \cong T_p$ (vectors)

  " $\binom{0}{0} = \mathbb{R}$ (numbers)

- tensor product

  $(T \otimes S)(\omega, X, Y) = T(\omega, X)\, S(Y)$

- tensors of type $\binom{p}{q}$ are linear combinations of tensors products of p vectors and q covectors

In components: $T = \varphi^* \bar{T}$ reads

$$T^i{}_j = (\varphi_*^{-1})^i{}_\alpha (\varphi^*)_j{}^\beta \bar{T}^\alpha{}_\beta$$

$$= \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\beta}{\partial x^j} \bar{T}^\alpha{}_\beta$$

canonical
bases

Adjoint map (pull back)

$$\varphi^*: T^*_{\bar{p}}(\bar{M}) \to T^*_p(M)$$
$$\bar{\omega} \mapsto \varphi^* \bar{\omega} \equiv \omega$$

$$\langle \varphi^* \bar{\omega}, X \rangle := \langle \bar{\omega}, \varphi_* X \rangle \qquad (X \in T_p(M))$$

Components

$$\omega_k = (\varphi^*)_k{}^i \, \bar{\omega}_i = (\varphi_*)^i{}_k \, \bar{\omega}_i$$

Extension to tensors on T_p:

Possible, if φ is invertible in a nbhd of p with φ^{-1} smooth (local diffeomorphism). Then φ_*, φ^* invertible

E.g. $\bar{T}$ of type $\binom{1}{1}$ on $T_{\bar{p}}$

$$(\varphi^* \bar{T})(\omega, X) = \bar{T}(\varphi^{*-1} \omega, \varphi_* X)$$

Properties

- $\varphi^*(\bar{T} \otimes \bar{S}) = (\varphi^* \bar{T}) \otimes (\varphi^* \bar{S})$
- $\mathrm{tr}\,(\varphi^* \bar{T}) = \varphi^*(\mathrm{tr}\,\bar{T})$ (any trace)
- $\varphi^* \bar{f} = \bar{f} \circ \varphi \qquad (\bar{f} \in \bar{\mathcal{F}})$

Vector fields

A vector field X on M is a linear map

$$X : \mathcal{F} \to \mathcal{F} \quad , \quad f \longmapsto Xf$$

smooth functions on M

with product rule

$$X(fg) = (Xf)g + f(Xg)$$

Fact: $(Xf)(p)$ depends only on f in an arbitrarily small nbhd of p.

Hence: for any $p \in M$

$$X_p : f \longmapsto (Xf)(p) \qquad (f \in \mathcal{F}_p)$$

defines $X_p \in T_p$.

In a chart:

$$X = X^i(x) \frac{\partial}{\partial x^i} \quad \text{with} \quad X^i = X x^i$$

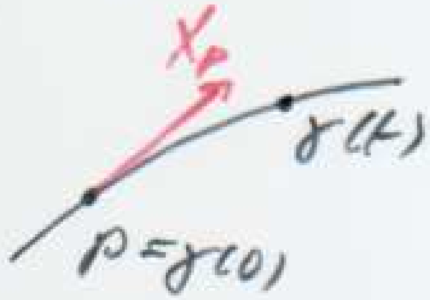
Flow on M: 1-parameter group of diffeos

$$\varphi_t : M \to M \quad (t \in \mathbb{R})$$

$$\varphi_t \circ \varphi_s = \varphi_{t+s}$$

with smooth orbits

$$t \mapsto \varphi_t(p) = \gamma(t) \,, \qquad (p \in M)$$



A flow determines a generating vector field X

$$X_p = \left.\frac{d\gamma}{dt}\right|_{t=0} ,$$

and viceversa

$$\frac{d\gamma}{dt} = X_{\gamma(t)} \quad , \quad \gamma(0) = p$$

X vector field on M, φ_t corresponding flow

<u>Def.</u> The <u>Lie derivative</u> $L_X R$ of a tensor field R in direction of X is

$$L_X R = \frac{d}{dt}\varphi_t^* R\Big|_{t=0}$$

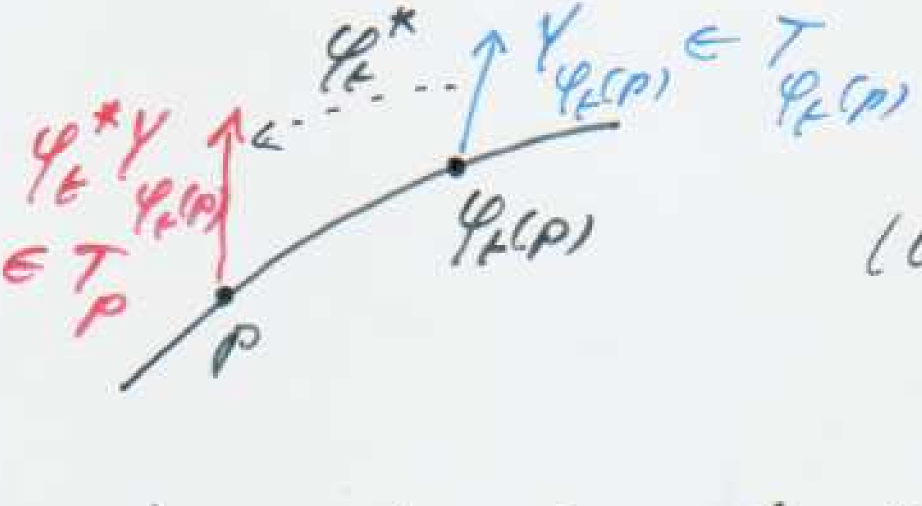
i.e.

$$(L_X R)_p = \frac{d}{dt}\varphi_t^* R_{\varphi_t(p)}\Big|_{t=0}$$

In components (R of type $\binom{1}{1}$)

$$(L_X R)^i_j = R^i_{j,k}X^k - R^\alpha_j X^i_{,\alpha} + R^i_\beta X^\beta_{,j}$$

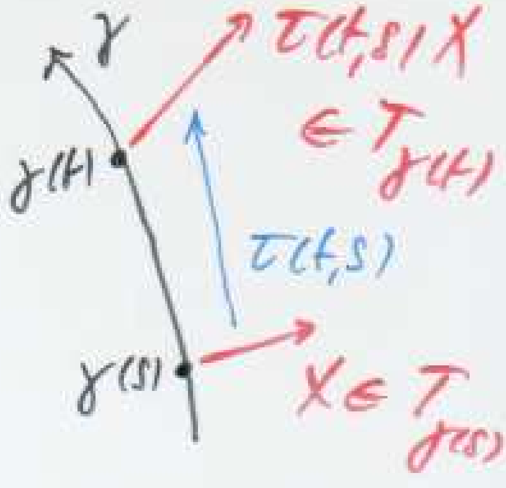
As a picture ($R = Y$ vector field)



(Lie transport)

$$(L_X Y)^i = Y^i_{,k}X^k - Y^k X^i_{,k}$$

Definition Along any curve γ in M a parallel transport of vectors is defined: a linear map



$$\tau(t,s): T_{\gamma(s)} \to T_{\gamma(t)}, \quad X \longmapsto \tau(t,s)X$$

with

1) $\tau(t,t) = 1 \quad , \quad \tau(t,s)\tau(s,r) = \tau(t,r)$

2) in any chart

$$\frac{\partial}{\partial t} \tau^i{}_k(t,s)\Big|_{t=s} = -\Gamma^i{}_{lk}(\gamma(s))\dot{\gamma}^l(s)$$

$\Gamma^i{}_{lk}(x)$: Christoffel symbols of the transports

Properties

- a parallel transported vector

$$X(t) = \tau(t,s)X(s)$$

obeys the ODE

$$\dot{X}^i(t) + \Gamma^i{}_{lk}(\gamma(t))\dot{\gamma}^l(t)X^k(t) = 0 \; ;$$

and viceversa.

$\dot{X}^i$ are derivatives w.r.t. t of components $X^i(t)$ of a vector $X(t) \in T_{\gamma(t)}$; but <u>not</u> themselves the components of a vector.

→ $\Gamma^i_{\ell k}$ are not the components of a tensor

- $\frac{\partial}{\partial s} \tau^i_j(t,s)\Big|_{s=t} = \Gamma^i_{\ell j}(\gamma(t))\dot{\gamma}^\ell(t)$