Exercise 8 – Fermionic Delta Functions

The integration over anti-commuting or Grassmann odd variables is defined by

$$\int d\theta 1 = 0, \quad \int d\theta \theta = 1,$$

where $\theta$ is an anti-commuting coordinate. I.e. integration is identical to differentiation here.

1) Show that $\delta(\theta) = \theta$ by integrating the fermionic $\delta$-function against a test-function $F(\theta)$.

2) Prove the following relations for the helicity spinors $\lambda$ and $\mu$

$$\delta^{(2)}(\lambda^\alpha a + \mu^\alpha b) = \begin{cases} \delta(a) \delta(b) |\langle\lambda\mu\rangle| & \text{for } a, b \text{ Grassmann even (commuting)}, \\ \delta(a) \delta(b) \langle\lambda\mu\rangle & \text{for } a, b \text{ Grassmann odd (anti-commuting)} \end{cases}$$

3) Use this to show that

$$\delta^{(8)} \left( \sum_{i=1}^{3} \tilde{\lambda}_i^\alpha \tilde{\eta}_{iA} \right) = [12]^4 \delta^{(4)} \left( \tilde{\eta}_{1A} - \frac{[23]}{[12]} \tilde{\eta}_{3A} \right) \delta^{(4)} \left( \tilde{\eta}_{2A} - \frac{[31]}{[12]} \tilde{\eta}_{3A} \right)$$

where $\tilde{\eta}_{iA}$ are anti-commuting variables. In fact they are the complex conjugates of the $\eta_{iA}$ that were introduced in class.

Exercise 9 – 3-point Superamplitudes

1) Argue that for MHV 3-particle kinematics, i.e. $[ij] = 0$ but $\langle ij \rangle \neq 0 \forall i,j \in \{1,2,3\}$, together with the conditions of $p$ and $q$ invariance and local helicity $h_i = 1$ the only possible form of the 3-point MHV amplitude is

$$A_{3}^{MHV} = \frac{\delta^{(4)}(\sum \lambda_i \tilde{\lambda}_i) \delta^{(8)}(\sum \lambda_i^\alpha \eta_i^A)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}.$$
2) This entails the complex conjugated relation

$$\overline{A}_3^{MHV} = (A_3^{MHV})^* = -\delta^{(4)} \left( \sum \lambda_i \bar{\lambda}_i \right) \delta^{(8)} \left( \sum \lambda_i^A \bar{\eta}_A \right) \delta^{(4)} \left( \bar{\eta} A_{[12]} + \bar{\eta} A_{[23]} + \bar{\eta} A_{[31]} \right),$$

using $\langle ij \rangle^* = -[ij]$. In order to transform this back into the $\{\lambda, \bar{\lambda}, \eta\}$ original on-shell superspace we need to perform a Fourier transformation of the anti-commuting $\bar{\eta}$ in the sense of

$$\Phi(\eta) = \int d^4 \bar{\eta} e^{i \eta^A \bar{\eta}_A} \Phi(\bar{\eta}).$$

Show that under this Fourier transformation using the result (1) one obtains the anti-MHV 3-point superamplitude

$$\overline{A}_3^{MHV} = -\delta^{(4)} \left( \sum \lambda_i \bar{\lambda}_i \right) \delta^{(4)} \left( \bar{\eta} A_{[12]} + \bar{\eta} A_{[23]} + \bar{\eta} A_{[31]} \right).$$