Physical interpretation of the ~~Hessian~~ effective action

Assume that we can have a static field configuration

$$\langle \phi(x) \rangle = \langle \Omega | \hat{\phi}(x) | \Omega \rangle$$

~~what~~ with $\frac{\partial}{\partial t} \langle \phi(x) \rangle = 0$.

The system is under the influence of a source $J(x)$. What is the minimum energy of the system for the given field configuration?

The energy is given by

$$\langle \Omega | H | \Omega \rangle = E[J]$$

and we have normalized

$$\langle \Omega | \Omega \rangle = 1.$$

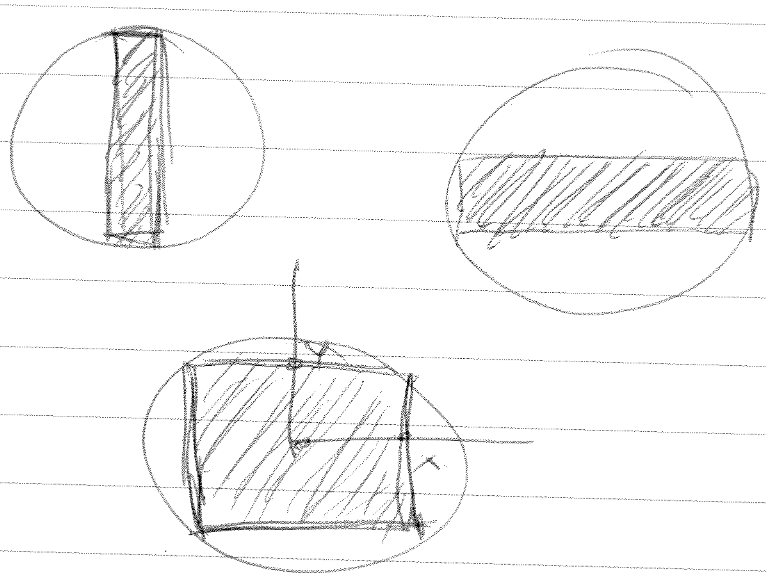
This is a minimization problem.

Minimization with constraints (Laplace multipliers)

Suppose we would like to solve the following problem:

What is the maximum surface of a rectangle which we can inscribe in a

Circle of radius R?



$$S = 4xy \qquad x^2+y^2=R^2$$

Minimize the Lagrangian:

$$L = S + \lambda\left(x^2+y^2-R^2\right)$$

↳ Lagrange multiplier.

$$= 4xy + \lambda\left(x^2+y^2-R^2\right)$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} = 0 &\rightarrow 4y + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = 0 &\rightarrow 4x + 2\lambda y = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow 4\left(x^2+y^2\right) + 4\lambda xy = 0 \Rightarrow$$

$$\Rightarrow \cancel{xy} \;\lambda = -\frac{R^2}{xy}$$

Substituting back

$$\rightarrow y - \frac{R^2}{2y} = 0 \Rightarrow y = \pm\frac{R}{\sqrt{2}}$$

Similarly $x = \pm R/\sqrt{2}$.

Can we do this in quantum mechanics? Assume a Hamiltonian H. What is the state of minimum energy?

We construct the Lagrangian

$$L = \underbrace{\langle 0 | H | 0 \rangle}_{\substack{\text{minimize} \\ \text{energy}}} - \lambda \underbrace{\langle 0 | 0 \rangle}_{\substack{\text{normalization} \\ \text{constraint on} \\ \text{quantum state}}}$$

~~wave of~~

$$= \int d^3\vec{x} \, \langle 0 | \vec{x} \rangle \langle \vec{x} | H - \lambda | 0 \rangle$$

$$= \int d^3\vec{x} \, \psi_0^*(x) \, \langle \vec{x} | H - \lambda | 0 \rangle$$

Varying with respect to $\dfrac{\delta}{\delta \psi_0^*(\vec{y})}$ gives

$$\frac{\delta L}{\delta \psi_0^*(\vec{y})} = 0 \Rightarrow \langle \vec{y} | H - \lambda | 0 \rangle \quad \forall \vec{y}.$$

Thus, $(H - \lambda)|0\rangle = 0$ ~~and~~ which is nothing else that Schrödinger's equation.

Let us now go to our field theory problem. What is the minimum energy $E[J]$ for a given field configuration

$$\langle 0 | \hat{\phi}(0) \rangle = \langle \phi(x) \rangle$$

which is static

We ~~start~~ create the "Lagrangian"

$$L = \langle \underline{0} | H | \underline{0} \rangle - \lambda (\langle \underline{0} | \underline{0} \rangle - 1)$$

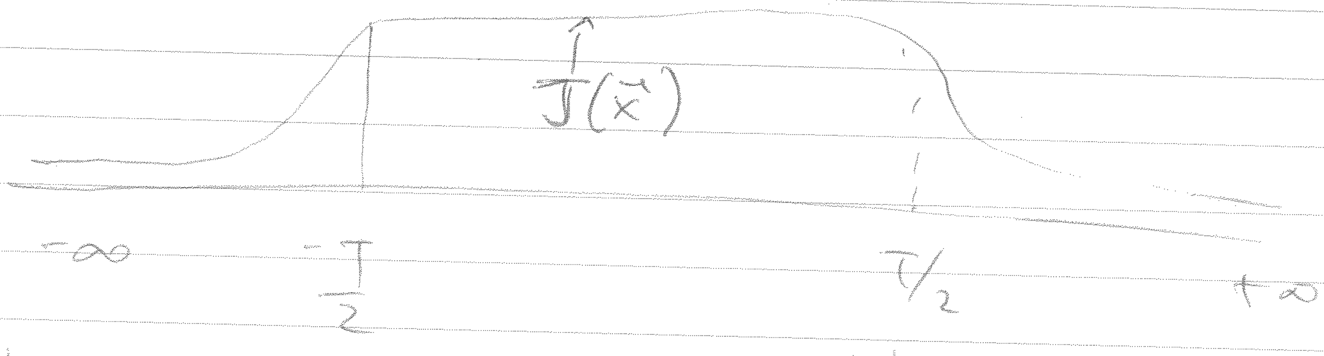
$$- \int d^3\vec{x}\, f(x) \left(\langle \underline{0} | \hat{\phi}(x) | \underline{0} \rangle - \langle \phi(x) \rangle \right)$$

This gives the solution:

$$\boxed{H | \underline{0} \rangle - \lambda | \underline{0} \rangle - \int d^3\vec{x}\, f(\vec{x})\, \hat{\phi}(\vec{x}) | \underline{0} \rangle = 0.}$$

Obviously, it is not easy to solve this equation. Let's use some physics intuition.

Assume a source ~~fed~~ $J(\vec{x}, t)$ ~~of the~~ with the following time dependence.



The source is switched on adiabatically, i.e. $\frac{\partial H}{\partial t} \equiv$ very small.

In the adiabatic approximation the system transitions ~~during~~ $t_1 \to t_2$ from the ground state of $H(t_1)$ to the ground state of $H(t_2)$, picking up a phase.

Exercise: Prove the above statement in quantum mechanics.

$$\langle 0, -\infty | 0, +\infty \rangle_J = e^{-i E[J] \cdot T}$$

$E[J]$ → energy, functional of the source.

But also:

$$\langle 0, -\infty | 0, +\infty \rangle_J = e^{+i W[J]}$$

Then

$$\boxed{W[J] = -E[J] \cdot T}$$

We have assumed that $E[J]$ is an eigenstate of the Hamiltonian when the source is switched on. The Hamiltonian is:

$$\left(H - \int d^3\vec{x}\, J(\vec{x})\, \Phi(x) \right)$$

and the system is in a state where:

$$\left[H - \int d^3\vec{x}\, J(x)\, \Phi(x) - E[J] \right] |\psi\rangle = 0$$

Assuming that the transition is adiabatic, $E[J]$ is a minimum energy.

Compare this equation with our minimization procedures

$$\left[H - \int d^3\vec{x}\, f(x)\, \hat{\phi}(x) - \lambda\right] |\underline{o}\rangle = 0$$

Then $f(\vec{x}) = J(\vec{x})$ ~~the~~

$$|\underline{o}\rangle = |\psi\rangle$$

and $\lambda = E[J]$

Thus $J(\vec{x})$ is the current which creates a field configuration

$$\langle \phi(\vec{x}) \rangle = \langle \underline{o} | \phi(\vec{x}) | \underline{o} \rangle$$ for

which the energy $E[J]$ is minimum.

Thus we have:

$$\langle \underline{o} | H | \underline{o} \rangle = -\frac{W[J]}{T} + \frac{\int d^4x\, J(\vec{x},t) \langle \phi(\vec{x}) \rangle}{T}$$

$$\Rightarrow \boxed{\langle \underline{o} | H | \underline{o} \rangle = -\frac{1}{T} \Gamma[\langle \phi \rangle]}$$

minimum energy.

For constant field configurations

$$\Gamma[\langle\phi\rangle] = -T \cdot (\text{Volume}) \cdot V_{eff}(\langle\phi\rangle)$$

$$\Rightarrow \frac{\langle 0|H|0\rangle}{(\text{Volume})} = V_{eff}(\langle\phi\rangle)$$

minimum energy
per unit volume.

The interpretation of V_{eff} as minimum energy per unit volume tells that without currents the vacuum state relaxes to not only a stationary value for V_{eff}, but to a true minimum of V_{eff}.

Concave effective potential

Recall that $\frac{\partial^2 V_{eff}}{\partial\langle\phi(x)\rangle\partial\langle\phi(x)\rangle}$ is related

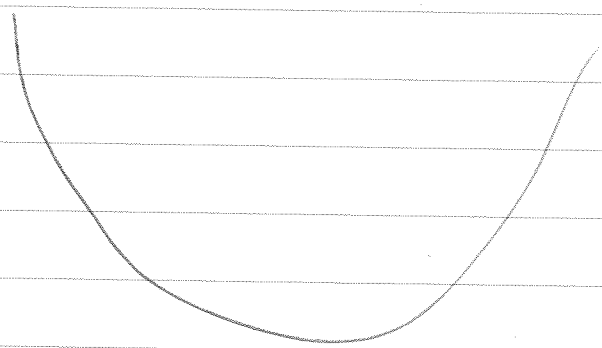
to the mass matrix of the theory.

The latter is positive definite!

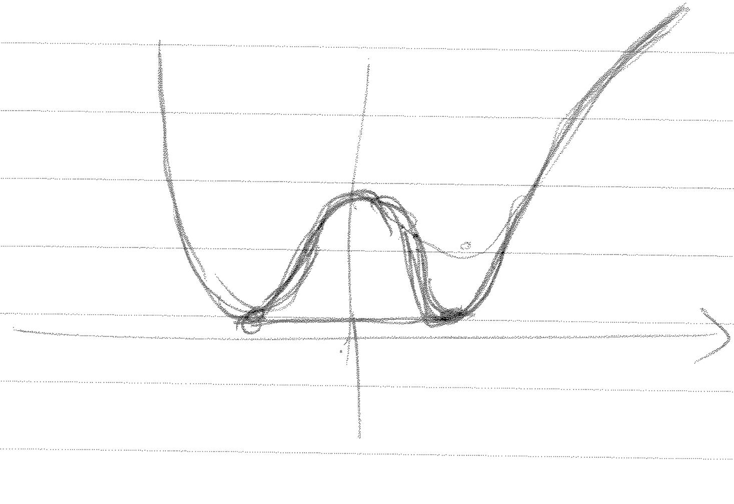
This implies that V_{eff} is **convex**

A proof is beyond what we can demonstrate in this class. But read:

J. Iliopoulos, C. Itzykson, A. Martin
Rev. Mod. Phys. 47 (1975) 165



What does this mean for a classical theory with a potential



How is V_{eff}?

We shall prove later that

$$\langle 0_1 | \hat{O} | 0_2 \rangle = 0 \quad \text{in quantum field theory!}$$

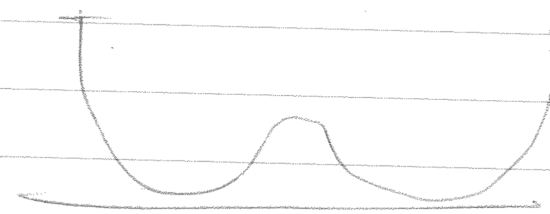
Then, we have that

$$\lambda = \sin^2\theta \qquad \bar{\lambda} = \cos^2\theta.$$

$$\langle 0 | H | 0 \rangle = \lambda E_1 + \bar{\lambda} E_2$$

$$\langle 0 | \phi | 0 \rangle = \lambda \langle \phi_1 \rangle + \bar{\lambda} \langle \phi_2 \rangle.$$

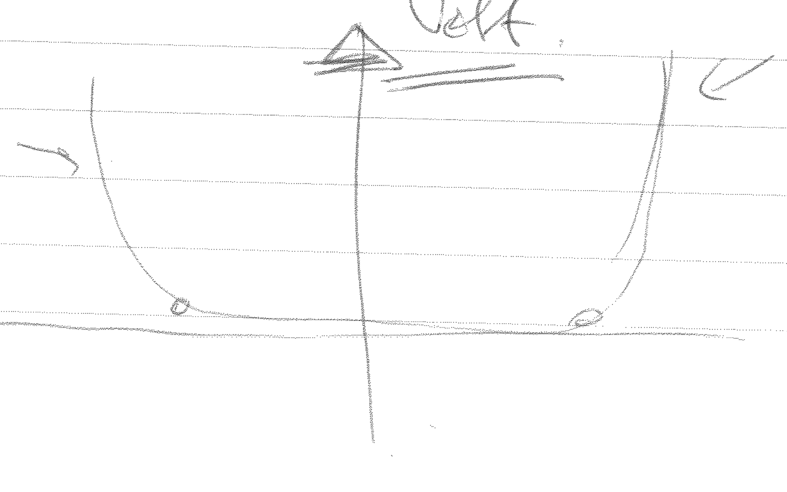
For $E_1 = E_2$.



$$\frac{\langle 0 | H | 0 \rangle}{\langle 0 | 0 \rangle} = E = \text{minimum} = V_{eff}(\langle \phi \rangle)$$

when $\langle \phi \rangle = \lambda \langle \phi_1 \rangle + \bar{\lambda} \langle \phi_2 \rangle$.

Thus



Proof that $\langle \underline{o}_1 | \hat{O} | \underline{o}_2 \rangle = 0$, where $\hat{O}$ Herm

This is due to the infinite volume of space. A ~~stat~~ vacuum state in infinite volume has zero momentum.

$$\hat{P} |v\rangle = 0$$

The vacuum state is part of the discreet spectrum. There are 1-particle & multi-particle states with zero momentum but the eigenvalues for them are continuous variables.

We can have many such vacuum states, which we can choose to be orthogonal:

$$\langle u | v \rangle \equiv \delta_{uv}$$

Now, take the matrix element for two equal time operators, separated by a space-like distance:

$$\langle u | A(\vec{x}) B(0) | v \rangle = \sum_{w} \langle u | A(\vec{x}) | w \rangle \langle w | B(0) | v \rangle$$

↳ discrete vacuum states

$$\cancel{= \sum_{w} \langle u | A(0) | w \rangle \langle w | B(0) | v \rangle}$$

$$+ \int d^3\vec{p} \sum_{N} \langle u | A(\vec{x}) | N, \vec{p} \rangle \langle N, \vec{p} | B(0) | v \rangle$$

continuum momentum states.

The commutator $[\hat{A}(\vec{x}), B(0)] = 0$ due to causality {$(0,0)$ & $(0,\vec{x})$ are separated by a space-like distance. Therefore:

$$0 = \langle u | [\hat{A}(x), \hat{B}(0)] | v \rangle \Rightarrow$$

$$\Rightarrow \sum_w \langle u | A(0) | w \rangle \langle w | B(0) | v \rangle$$

$$= \sum_w \langle u | B(0) | w \rangle \langle w | B(0) | v \rangle$$

The Hermitian matrices

$$\left[\langle u | A(0) | v \rangle \right]_{uv}$$

$$\left[\langle u | B(0) | v \rangle \right]_{uv}$$

commute with each other and can be diagonalized simultaneously. In this basis:

$$\langle u | A(0) | v \rangle = a_v \, \delta_{uv} \qquad a_v \in \mathbb{R}$$

$$\rightsquigarrow \quad \langle u | A(0) | v \rangle = 0 \quad \text{if } |u\rangle \neq |v\rangle.$$

We have proven that any **Hermitian** operator, yields vanishing matrix-elements for different vacuum states. It also ~~follows~~ ~~for~~ applies for the Hamiltonian

$$\langle a | H | b \rangle = 0 \quad , \text{ but also}$$

for small perturbations

$$\langle u|\delta H|v\rangle = 0.$$

Therefore, once in a vacuum state there is a zero probability to transition to ~~another~~ a ~~two~~ different vacuum state (of equal total energy)! But we can roll from one minimum to another.