Tenth Exercise Sheet due to 22. May

Recently, much attention was devoted to classify non-Markovianity of quantum evolutions Λ_t, $t \geq 0$. We explore the problem here; Λ_t is a family of CP-maps describing the evolution of a state ρ of S from time 0 to time t, $\rho_t = \Lambda_t \rho_0$.

Definition 1 (Markovianity) *We say that Λ_t is a Markovian evolution if there exists a family of CP-maps $V_{t,s}$ (propagators) such that*

$$\Lambda_t = V_{t,s}\Lambda_s \quad \textit{for all} \quad t \geq s.$$

Exercise 1 *Why is it a reasonable definition of Markovianity?*

Exercise 2 *Prove that $V_{t,s}$ satisfies the composition law*

$$V_{t,s}V_{s,u} = V_{t,u} \quad \textit{for all} \quad t \geq s \geq u.$$

Exercise 3 *Prove that if Λ_t is Markovian than*

$$S(\Lambda_t\rho_1|\Lambda_t\rho_2) \leq S(\Lambda_s\rho_1|\Lambda_s\rho_2) \quad \textit{for all} \quad t \geq s, \tag{1}$$

where S is the relative entropy and ρ_1, ρ_2 are arbitrary states.

Exercise 4 *Consider an evolution of a two level system*

$$\dot{\rho}_t = L_t\rho, \quad L_t\rho := \frac{1}{2}\gamma(t)(\sigma_z\rho\sigma_z - \rho)$$

and the corresponding family Λ_t. Show that

(i) *Λ_t are CP maps if and only if $\int_0^t \gamma(s)\mathrm{d}s \geq 0$ for all $t \geq 0$,*

(ii) *Λ_t is Markovian if and only if $\gamma(t) \geq 0$ for all $t \geq 0$,*

(iii) *The relation (1) cease to hold if Λ_t is not Markovian. This was in fact suggested as on of the possible measures of non-Markovianity.*

You can compute Λ_t explicitly and check all the assertions. Alternatively you can understand the assertions by drawing evolution Λ_t on the Bloch sphere.