Viscosity

We would like to describe phenomena related to dissipation in the fluid. This occurs due to irreversible processes. To describe them we should modify equations of motion of an ideal fluid. From the derivation it is clear, that the continuity equation remains unchanged. The Euler's equation, however, should be modified.

As we derived previously (momentum flux) the Euler's equation can be written as

$$\frac{\partial}{\partial t} \rho v_i = -\frac{\partial \Pi_{ik}}{\partial x_k} \quad , \quad \Pi_{ik} = p\,\delta_{ik} + \rho v_i v_k$$

To describe internal friction one should add to the "ideal" momentum flux a viscous stress tensor σ'_{ik}

$$\Pi_{ik} = p\,\delta_{ik} + \rho v_i v_k - \sigma'_{ik}$$

One can relate σ'_{ik} with velocity $v(r)$ using the following consideration. Internal friction appears only when different parts of fluid move with different velocities $\Rightarrow$

$\sigma'_{ik} = 0$ if $v(r) = \text{const}$, thus σ'_{ik} should depend on spatial derivatives of velocity.

For small gradients one assumes that this dependence is linear. Also σ'_{ik} should be zero if fluid is rotated as a whole $\vec{v} = \vec{\Omega} \times \vec{r}$. Thus σ'_{ik} should depend only on symmetric combination

$$\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}$$

Most general expression for this is

$$\sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) + \eta' \delta_{ik} \frac{\partial v_e}{\partial x_e}$$

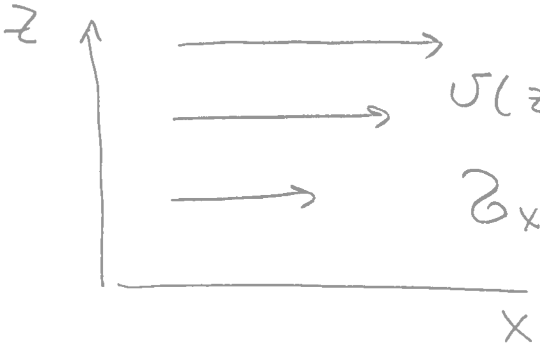
Compare with the Hooke's low $\sigma_{ik} \propto u_{ik}$

Sometimes one rewrites it in a different form

$$\sigma_{ik} = \underbrace{\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)}_{\text{traceless part}} + \xi\,\delta_{ik}\frac{\partial v_l}{\partial x_l}$$

η is <u>viscosity</u> coefficient

ξ is second <u>viscosity</u> , $\eta, \xi > 0$



$$\sigma_{xz} = \eta\frac{\partial v}{\partial z}$$

Substituing σ'_{ik} into the Euler's equation

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial}{\partial x_k}\Big[p\delta_{ik} - \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \eta'\delta_{ik}\frac{\partial v_l}{\partial x_l}$$

In general $\eta(p,\rho)$ and $\eta'(p,\rho)$ can depend on coordinate. Assuming the variations of p, ρ be small we put η, μ constant.

Then we get the famous Navier-Stokes equation (NSE) C. L. Navier 1827, G. G. Stokes 1845)

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\vec{\nabla})\vec{v}\right)=-\nabla p+\eta\nabla^2\vec{v}+(\eta+\eta')\,\mathrm{grad}\,\mathrm{div}\,\vec{v}$$

For incompressible fluid it is simplified

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\vec{\nabla})\vec{v}=-\frac{\mathrm{grad}\,p}{\rho}+\frac{\eta}{\rho}\nabla^2\vec{v}$$

$$\sigma_{ik}=-p\,\delta_{ik}+\eta\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)$$

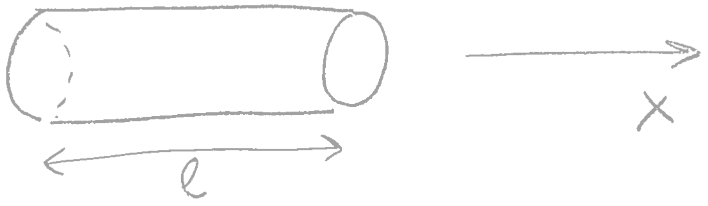
$\nu=\dfrac{\eta}{\rho}$ is called kinematic viscosity

	η $\frac{g}{cm\cdot sec}$	ν $\frac{cm^2}{sec}$
Water	0.01	0.01
Air	$1.8\cdot 10^{-4}$	0.15
Glycerine	8.5	6.8

Note that $\nu_{air} > \nu_{water}$

Navier-Stokes equation has higher-order derivatives (second) than Euler's equation so that we need more boundary conditions. Since we accounted the forces between fluid layers, we have also to account for the forces of molecular attraction between a viscous fluid and a solid body surface. Such force makes the layer of adjacent fluid to stick to the surface: $v=0$ on the surface (not only $v_n=0$ as for Euler). Note that the solutions of Euler's equation do not generally satisfy such boundary conditions. That means that even very small viscosity must play a role near a solid surface

Viscous flow in a pipe



$v_x(y,z) \Rightarrow \operatorname{div} v = 0$

y and z components of NSE give

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \Rightarrow p = p(x)$$

x component gives

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\eta}\frac{dp}{dx} \Rightarrow \frac{dp}{dx} = \text{const}$$

Thus we have $\nabla^2 v = \text{const}$
with boundary condition $v = 0$

For the pipe with circular cross-section

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\frac{\Delta p}{\eta \ell}$$

Then

$$v = -\frac{\Delta P}{4\eta l} r^2 + a \ln r + b$$

$a = 0$ because v is finite at the center of the pipe.

b is determined from the bounary condition

$v(R) = 0$, R - radius of the pipe.

$$v = \frac{\Delta P}{4\eta l}\left(R^2 - r^2\right)$$

The flux through the tube per unit time is

$$Q = \rho \int v \, d^2 r \quad \Rightarrow$$

$$Q = \frac{\pi \Delta P}{8 \nu l} R^4$$

Empirically G. Hagen (1839), J.L.M. Poiseuille 1840

explained by Stokes 1845