Exercise 1. ***Irreducible Tensor Operators: Wigner-Eckart Theorem.***

Tensor operators can be seen as a set of operators that collectively transform into each other in a specific way under rotations of space. (For example, the components of a vector observable transform into each other under rotations like those of a usual vector in $\mathbb{R}^3$.)

Formally, a collection of $2k+1$ operators $T_q^{(k)}$ (with $k \geqslant 0$ integer and $q = -k, -k+1, \ldots, k$) form an *irreducible tensor operator of rank* k if they satisfy the following commutation relations with the total angular momentum $\boldsymbol{J}$ of the physical system:

$$[J_z, T_q^{(k)}] = \hbar\, q\, T_q^{(k)} \ , \tag{1}$$

$$[J_+, T_q^{(k)}] = \hbar\, \sqrt{k(k+1) - q(q+1)}\, T_{q+1}^{(k)} \ , \tag{2}$$

$$[J_-, T_q^{(k)}] = \hbar\, \sqrt{k(k+1) - q(q-1)}\, T_{q-1}^{(k)} \ . \tag{3}$$

The *Wigner-Eckart theorem*, which will be proven in this exercise, states that the matrix elements of any such tensor operator are actually proportional to the Clebsch-Gordon coefficients. That is, if $\{|n, j, m\rangle\}$ is a standard basis of common eigenstates to $\boldsymbol{J}^2$ and J_z,

$$\langle n, j, m | T_q^{(k)} | n', j', m' \rangle = \alpha \cdot \langle j', k; m', q | j, m \rangle \ , \tag{4}$$

where the proportionality constant α only depends on n, j, k, n', j' (and not q, m, m') and is usually written in the form:

$$\alpha = \frac{1}{\sqrt{2j+1}} \left\langle n, j \middle\| \, T^{(k)} \, \middle\| n', j' \right\rangle \ . \tag{5}$$

(a) Show that a scalar observable is an irreducible tensor operator of rank $k = 0$, and that the three standard components of a vector observable are the components of an irreducible tensor operator of rank $k = 1$. Show that the angular momentum operators J_x, J_y, J_z are themselves a vector observable.

The *standard components* of a vector observable $\boldsymbol{V}$ are

$$V_1^{(1)} = -\frac{1}{\sqrt{2}} (V_x + iV_y) \tag{6}$$

$$V_0^{(1)} = V_z \tag{7}$$

$$V_{-1}^{(1)} = \frac{1}{\sqrt{2}} (V_x - iV_y) \tag{8}$$

Hint. A scalar *observable is invariant under rotations. A* vector *observable* $\boldsymbol{V}$ *transforms like a vector under rotations of space, i.e.* $\boldsymbol{V}' = R^{-1}\boldsymbol{V}$. *Consider infinitesimal rotations and remember that the rotations are generated by the* J *operators.*

Solution. An observable generally transforms under a rotation R as

$$A' = RAR^\dagger \ . \tag{L.1}$$

(We have simplified the notation by identifying the unitary action of the group element on the observable to the group element R itself.)

Recall that rotations of space (by an angle α about an axis $\boldsymbol{u}$) are generated by the angular momentum operators,

$$R = \mathrm{e}^{-\frac{i}{\hbar}\,\alpha\,\boldsymbol{J}\cdot\boldsymbol{u}} \ , \tag{L.2}$$

which for infinitesimal $\mathrm{d}\alpha$ yields

$$R_{\boldsymbol{u}}(\mathrm{d}\alpha) \approx \mathbb{1} - \frac{i}{\hbar}\mathrm{d}\alpha\,\boldsymbol{J}\cdot\boldsymbol{u} \ . \tag{L.3}$$

Equation (L.1) then becomes

$$A' \approx \left(\mathbb{1} - \frac{i}{\hbar}\mathrm{d}\alpha\,\boldsymbol{J}\cdot\boldsymbol{u}\right) A \left(\mathbb{1} + \frac{i}{\hbar}\mathrm{d}\alpha\,\boldsymbol{J}\cdot\boldsymbol{u}\right) \approx A - \frac{i}{\hbar}\mathrm{d}\alpha\,[\boldsymbol{J}\cdot\boldsymbol{u}, A] \ . \tag{L.4}$$

Now consider a scalar observable A, i.e. $A' = A$ for any rotation $R_{\boldsymbol{u}}(\mathrm{d}\alpha)$. Appropriate choices of $\boldsymbol{u}$ in (L.4) give respectively

$$[J_x, A] = 0 \ , \quad [J_y, A] = 0 \quad [J_z, A] = 0 \ , \tag{L.5}$$

which in turn implies, by definition of $J_\pm = J_x \pm iJ_y$,

$$[J_+, A] = 0 \ , \qquad [J_-, A] = 0 \ . \tag{L.6}$$

Let $\boldsymbol{V}$ a vector observable, that is, for any rotation R,

$$\boldsymbol{V}' = R^{-1}\boldsymbol{V} \ . \tag{L.7}$$

(this R is the 3×3 matrix that rotates the components of the vector, not to be confused with the rotation operator (L.2) that acts on operators.)

Consider an infinitesimal rotation about the Z axis and look at V_x. On one hand side, eq. (L.1) applies, giving thanks to (L.4),

$$V_x' \approx V_x - \frac{i}{\hbar}\mathrm{d}\alpha[J_z, V_x] \ ; \tag{L.8}$$

however on the other hand the three components of $\boldsymbol{V}$ have to transform according to (L.7), giving

$$V_x' \approx V_x + \mathrm{d}\alpha V_y \ . \tag{L.9}$$

We see from these two equations that (with $\mathrm{d}\alpha \to 0$)

$$[J_z, V_x] = i\hbar V_y \ . \tag{L.10}$$

A similar consideration with rotations about other axes reveal more generally

$$[J_i, V_j] = i\hbar\,\varepsilon_{ijk}V_k \ . \tag{L.11}$$

This can also be derived directly generally by writing (L.9) for a general rotation as $\boldsymbol{V} \approx \boldsymbol{V} + \mathrm{d}\alpha\,(\boldsymbol{u}\times\boldsymbol{V})$ and comparing with (L.4).

Let's consider condition (1). Using (L.11),

$$[J_z, V_0^{(1)}] = [J_z, V_z] = 0 \ , \tag{L.12}$$

$$[J_z, V_{\pm 1}^{(1)}] = \frac{1}{\sqrt{2}}\Big(\underbrace{[J_z, V_x]}_{i\hbar V_y} \pm i\underbrace{[J_z, V_y]}_{-i\hbar V_x}\Big) = \pm\hbar\,V_{\pm 1}^{(1)} \ . \tag{L.13}$$

$$\tag{L.14}$$

Similarly, condition (3) for $V_\pm^{(1)}$ becomes

$$[J_-, V_\pm^{(1)}] = \frac{1}{\sqrt{2}}\big[J_x - iJ_y, V_x \pm iV_y\big] = \frac{1}{\sqrt{2}}\Big(\pm i\underbrace{[J_x, V_y]}_{i\hbar V_z} - i\underbrace{[J_y, V_x]}_{-i\hbar V_z}\Big) = \frac{\hbar}{\sqrt{2}}\left(\pm(-V_z) - V_z\right) \ , \tag{L.15}$$

which correctly yields $[J_-, V_+^{(1)}] = \hbar\sqrt{2}V_0$ and $[J_+, V_+^{(1)}] = 0$; for $V_0^{(1)}$ this condition gives us

$$[J_-, V_0^{(1)}] = \underbrace{[J_x, V_z]}_{-i\hbar V_y} - i\underbrace{[J_y, V_z]}_{i\hbar V_x} = \hbar(V_x - iV_y) = \hbar\sqrt{2}\,V_-^{(1)} \ . \tag{L.16}$$

Similar considerations show the condition (2).

We have seen that a vector observable $\boldsymbol{V}$ satisfies the commutation relations (L.11). Conversely, we will show that any triplet of observables $\boldsymbol{V}$ that satisfies those commutation relations is a vector observable.

Write thanks to (L.4)

$$R\,V_j\,R^\dagger \approx V_j - \frac{i}{\hbar}\,\mathrm{d}\alpha \sum_i u_i\,[J_i, V_j] = V_j + \mathrm{d}\alpha\,u_i\,\varepsilon_{ijk} V_k = V_j - \mathrm{d}\alpha\,(\boldsymbol{u}\times\boldsymbol{V})_j \;, \approx \left[R^{-1}\,\boldsymbol{V}\right]_j \;, \tag{L.17}$$

which is an infinitesimal rotation of the components of $\boldsymbol{V}$ about the axis $\boldsymbol{u}$. (On the left hand side of (L.17), R is the operator given by (L.2), wheras on the right hand side R is the 3×3 matrix representing the rotation of the components of the vector.) This shows that any triplet of observables satisfying the commutation relations (L.11) is a vector observable. In particular, the J operators themselves are a vector observable.

Alternatively to (L.17) one could have written

$$R\,V_j\,R^\dagger \approx V_j - \frac{i}{\hbar}\,\mathrm{d}\alpha \sum_i u_i\,[J_i, V_j] = V_j + \mathrm{d}\alpha\,u_i\,\varepsilon_{ijk} V_k = \left[(1-\mathrm{d}\alpha\,\boldsymbol{T}\cdot\boldsymbol{u})\,\boldsymbol{V}\right]_j \approx \left[R^{-1}\,\boldsymbol{V}\right]_j \;, \tag{L.18}$$

where $\boldsymbol{T}$ are the triplet of 3×3 matrices which generate the $SO(3)$ rotations, $(T_j)_{ik} = -\varepsilon_{jik}$ and $R = \mathrm{e}^{\mathrm{d}\alpha\,\boldsymbol{T}\cdot\boldsymbol{u}}$.

(b) An alternate definition for a collection of operators $T_q^{(k)}$ to be an irreducible tensor operator is that they transform under rotations as

$$R\,T_q^{(k)}\,R^\dagger = \sum_{q'} D_{q'q}^{(k)} T_{q'}^{(k)} \;, \tag{9}$$

where $D_{m'm}^{(j)} = \langle j, m'|R|j, m\rangle$ are the matrix elements of the rotation in the standard angular momentum basis.

Show that objects $T_q^{(k)}$ that transform as (9) fulfill the commutation relations (1)–(3).

Solution. Consider a collection of operators $T_q^{(k)}$ that transform according to (9). For a rotation by an infinitesimal angle $\mathrm{d}\alpha$ about an axis $\boldsymbol{u}$, we have

$$R\,T_q^{(k)}\,R^\dagger = \sum_{q'} D_{q'q}^{(k)}\,T_{q'}^{(k)} \approx \sum_{q'} \langle k\,q'|\Big(1 - \frac{i}{\hbar}\,\mathrm{d}\alpha\,\boldsymbol{J}\cdot\boldsymbol{u}\Big)|k\,q\rangle\,T_{q'}^{(k)} = T_q^{(k)} - \frac{i}{\hbar}\,\mathrm{d}\alpha \sum_{q'} \langle k\,q'|\boldsymbol{J}\cdot\boldsymbol{u}|k\,q\rangle\,T_{q'}^{(k)} \;, \tag{L.19}$$

where we have used the fact that $R = \mathrm{e}^{-\frac{i}{\hbar}\,\mathrm{d}\alpha\,\boldsymbol{J}\cdot\boldsymbol{u}}$.

On the other hand, we have from (L.4) that

$$R\,T_q^{(k)}\,R^\dagger \approx T_q^{(k)} - \frac{i}{\hbar}\,\mathrm{d}\alpha\,[\boldsymbol{J}\cdot\boldsymbol{u}, T_q^{(k)}] \;, \tag{L.20}$$

and by comparing with (L.19) and in the limit $\mathrm{d}\alpha \to 0$,

$$[\boldsymbol{J}\cdot\boldsymbol{u}, T_q^{(k)}] = \sum_{q'} \langle k\,q'|\boldsymbol{J}\cdot\boldsymbol{u}|k\,q\rangle\,T_{q'}^{(k)} \;. \tag{L.21}$$

Choose first $\boldsymbol{u}$ along the Z axis to obtain

$$[J_z, T_q^{(k)}] = \sum_{q'} \langle k\,q'|J_z|k\,q\rangle\,T_{q'}^{(k)} = \sum_{q'} \hbar\,q\,\langle k\,q'|k\,q\rangle\,T_{q'}^{(k)} = \sum_{q'} \hbar\,q\,\delta_{qq'}\,T_{q'}^{(k)} = \hbar\,q\,T_q^{(k)} \;. \tag{L.22}$$

Remember that $J_\pm = J_x \pm iJ_y$. Taking a linear combination of (L.21) with $\boldsymbol{u}$ respectively along the X and Y axes,

$$\begin{aligned}[J_\pm, T_q^{(k)}] = [J_x, T_q^{(k)}] \pm i[J_y, T_q^{(k)}] &= \sum_{q'} \langle k\,q'|\,(J_x \pm iJ_y)\,|k\,q\rangle\,T_{q'}^{(k)} = \sum_{q'} \langle k\,q'|J_\pm|k\,q\rangle\,T_{q'}^{(k)} \\ &= \sum_{q'} \langle k\,q'|k\,q{\pm}1\rangle \sqrt{k(k+1)-q(q\pm 1)}\,T_{q'}^{(k)} = \sqrt{k(k+1)-q(q\pm 1)}\,T_{q\pm 1}^{(k)} \;,\end{aligned} \tag{L.23}$$

showing that (2) and (3) are equally fulfilled.

(c) Let $T_q^{(k)}$ be the components of an irreducible tensor operator of rank k. Using (1), show that $\langle n, j, m|T_q^{(k)}|n', j', m'\rangle$ is zero if m is not equal to $q + m'$.

Solution. By (1) we must have

$$\hbar q\langle n,j,m|T_q^{(k)}|n',j',m'\rangle = \langle n,j,m|[J_z,T_q]|n',j',m'\rangle = \langle n,j,m|\left(\hbar m T_q^{(k)} - T_q^{(k)}\hbar m'\right)|n',j',m'\rangle \ , \quad \text{(L.24)}$$

such that

$$\hbar\langle n,j,m|T_q^{(k)}|n',j',m'\rangle\left(q-m+m'\right) = 0 \ . \quad \text{(L.25)}$$

These matrix elements may only be nonzero if $m = q + m'$.

(d) Proceeding in the same way with relations (2) and (3), show that the $(2j+1)(2k+1)(2j'+1)$ matrix elements $\langle n,j,m|T_q^{(k)}|n',j',m'\rangle$ corresponding to fixed values of n,j,k,n',j' satisfy recurrence relations identical to those satisfied by the $(2j+1)(2k+1)(2j'+1)$ Clebsch-Gordan coefficients $\langle j',k;m',q|j,m\rangle$ corresponding to fixed values of j,k,j'.

Hint. Recall the recursion relations for the Clebsch-Gordan coefficients for the addition of two angular momenta $|j_1,m_1\rangle$ *and* $|j_2,m_2\rangle$ *into a global angular momentum* $|J,M\rangle$*,*

$$\begin{aligned}\sqrt{J(J+1)-M(M-1)}\,&\langle j_1,j_2;m_1,m_2|J,M-1\rangle \\ &= \sqrt{j_1(j_1+1)-m_1(m_1+1)}\,\langle j_1,j_2;m_1+1,m_2|J,M\rangle \\ &+\sqrt{j_2(j_2+1)-m_2(m_2+1)}\,\langle j_1,j_2;m_1,m_2+1|J,M\rangle \ , \end{aligned} \quad (10)$$

$$\begin{aligned}\sqrt{J(J+1)-M(M+1)}\,&\langle j_1,j_2;m_1,m_2|J,M+1\rangle \\ &= \sqrt{j_1(j_1+1)-m_1(m_1-1)}\,\langle j_1,j_2;m_1-1,m_2|J,M\rangle \\ &+\sqrt{j_2(j_2+1)-m_2(m_2-1)}\,\langle j_1,j_2;m_1,m_2-1|J,M\rangle \ . \end{aligned} \quad (11)$$

Solution. Looking at matrix elements of (2),

$$\begin{aligned}\hbar\sqrt{k(k+1)-q(q+1)}\,\langle n,j,m|T_{q+1}^{(k)}|n',j',m'\rangle &= \langle n,j,m|[J_+,T_q^{(k)}]|n',j',m'\rangle \\ &= \hbar\,\langle n,j,m-1|\left(\sqrt{j(j+1)-m(m-1)}\,T_q^{(k)}\right)|n',j',m'\rangle \\ &- \hbar\,\langle n,j,m|\left(T_q^{(k)}\sqrt{j'(j'+1)-m'(m'+1)}\right)|n',j',m'+1\rangle \ , \end{aligned} \quad \text{(L.26)}$$

where we have used $J_+ = J_-^\dagger$, along with

$$J_\pm|n,j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|n,j,m\pm 1\rangle \ . \quad \text{(L.27)}$$

Equation (L.26) leads to

$$\begin{aligned}\sqrt{j(j+1)-m(m-1)}\,&\langle n,j,m-1|T_q^{(k)}|n',j',m'\rangle \\ &= \sqrt{k(k+1)-q(q+1)}\,\langle n,j,m|T_{q+1}^{(k)}|n',j',m'\rangle \\ &+ \sqrt{j'(j'+1)-m'(m'+1)}\,\langle n,j,m|T_q^{(k)}|n',j',m'+1\rangle \ , \end{aligned} \quad \text{(L.28)}$$

which are the same as the recursion relations for the Clebsch-Gordan coefficients (10).

Similarly, by looking at matrix elements of (3) we obtain relations similar to (11),

$$\begin{aligned}\sqrt{j(j+1)-m(m+1)}\,&\langle n,j,m+1|T_q^{(k)}|n',j',m'\rangle \\ &= \sqrt{k(k+1)-q(q-1)}\,\langle n,j,m|T_{q-1}^{(k)}|n',j',m'\rangle \\ &+ \sqrt{j'(j'+1)-m'(m'-1)}\,\langle n,j,m|T_q^{(k)}|n',j',m'-1\rangle \ , \end{aligned} \quad \text{(L.29)}$$

(e) Show that:

$$\langle n,j,m|T_q^{(k)}|n',j',m'\rangle = \alpha\cdot\langle j',k;m',q|j,m\rangle \ ,$$

for some α depending only on n,j,k,n',j'.

Solution. We know that the matrix elements $\langle n, j, m | T_q^{(k)} | n', j', m' \rangle$ behave essentially like Clebsch-Gordan coefficients, namely, they satisfy recursion relations (L.28) and (L.29), and to be nonzero they must satisfy

$$m = q + m' \ . \tag{L.30}$$

The idea here is to show that once we have fixed the value of the "first" matrix element, then all the others are proportionally determined by the recursion relations.

First consider the case where $m = j$. Equation (L.29) then becomes

$$\begin{aligned} 0 = \sqrt{k(k+1) - q(q-1)}\, \langle n, j, m = j | T_{q-1}^{(k)} | n', j', m' \rangle \\ + \sqrt{j'(j'+1) - m'(m'-1)}\, \langle n, j, m = j | T_q^{(k)} | n', j', m' - 1 \rangle \ , \end{aligned} \tag{L.31}$$

such that

$$\langle n, j, j | T_q^{(k)} | n', j', m' - 1 \rangle = -\frac{\sqrt{k(k+1) - q(q-1)}}{\sqrt{j'(j'+1) - m'(m'-1)}} \langle n, j, j | T_{q-1}^{(k)} | n', j', m' \rangle \ . \tag{L.32}$$

In addition, equation (L.30) forces $j = q + m' - 1$, i.e. $m' = j - q + 1$:

$$\langle n, j, j | T_q^{(k)} | n', j', j - q \rangle = -\frac{\sqrt{k(k+1) - q(q-1)}}{\sqrt{j'(j'+1) - m'(m'-1)}} \langle n, j, j | T_{q-1}^{(k)} | n', j', j - q + 1 \rangle \ . \tag{L.33}$$

This means that all matrix elements with $m = j$ (spanned by $q = -k, \dots, k$) are uniquely determined by the value of the first matrix element $\langle n, j, j | T_{-k}^{(k)} | n', j', j + k \rangle$.

It is now left to show that all other matrix elements for general $m \neq j$ are determined in a similar manner by the same matrix element $\langle n, j, j | T_{-k}^{(k)} | n', j', j + k \rangle$. This can be seen with the recursion relation (L.28). This relation allows to calculate any matrix element with a value of $m = j - 1$ by using only matrix elements with $m = j$. Applying this relation recursively, one can uniquely determine all matrix elements with $m = j - 2, j - 3 \dots$

Since the recursion relations are linear, the matrix elements depend linearly on the value of the "first" matrix element. The Clebsch-Gordan coeffients $\langle j', k; m', q | j, m \rangle$ themselves can be obtained in this manner by a suitable choice of the "first" coefficient. It follows that the matrix elements of $T_q^{(k)}$ have to be proportional to the Clebsch-Gordan coefficients.

(f) Use the Wigner-Eckart theorem to show that the state $\hat{\boldsymbol{r}} | n, l, m \rangle$ (where $\hat{\boldsymbol{r}} = \sum_j \hat{\boldsymbol{r}}_j$) lives in the subspace of representations $l - 1$, l, and $l + 1$ of $SU(2)$. This is used in Section 2.3 of the script.

Hint. Consider the dipole moment operators

$$D_{+1}^{(1)} = -\frac{1}{\sqrt{2}} \sum_j \hat{x}_j + i \hat{y}_j \ , \tag{12}$$

$$D_0^{(1)} = \sum_j \hat{z}_j \ , \tag{13}$$

$$D_{-1}^{(1)} = \frac{1}{\sqrt{2}} \sum_j \hat{x}_j - i \hat{y}_j \ . \tag{14}$$

Solution. The dipole moment operators are obviously the standard components of the vector operator $\hat{\boldsymbol{r}}$. They thus form an irreducible tensor operator of rank $k = 1$.

Matrix elements $\langle n', l', m' | D_q^{(1)} | n, l, m \rangle$ are given by the Wigner-Eckart theorem (4), and thus non-zero matrix elements may only be those for which the Clebsch-Gordan coefficient $\langle l, k = 1; m, q | l', m' \rangle$ does not vanish. This is only the case if $l' = l - 1, l, l + 1$.

Matrix elements $\langle n', l', m' | \hat{\boldsymbol{r}} | n, l, m \rangle$ are just appropriate linear combinations of the same matrix elements of the different $D_q^{(1)}$'s, so they also vanish if $l' \neq l - 1, l, l + 1$.