Üb 4.2

Conserved quantities

We have the mechanical system (Γ, ω) with the phase space Γ (symplectic manifold) and the symplectic structure ω. We assume that the vector field $J\,dH$ corresponding to the function H (on Γ) gives a one-parameter group of canonical transformations $\phi_t : \Gamma \to \Gamma$ through

$$J\,dH(x) = \frac{d}{dt}\Big|_{t=0} \phi_t(x) \qquad \forall x \in \Gamma, \quad \forall t \in I$$

We call $\{\phi_t\}_{t\in I}$ the Hamiltonian phase flow with Hamiltonian function H.

In the following we will also write $X_H \equiv J\,dH$ the vector field on Γ corresponding to the differential dH through the mapping $J : T_x^*\Gamma \to T_x\Gamma$.

The mapping J is an isomorphism between 1-forms and vector field which can be deffined through

$$\omega(\eta, J\alpha) = \alpha(\eta) \qquad \forall \eta \in T_x\Gamma, \quad \forall \alpha \in T_x^*\Gamma.$$

We can then write

$$\omega(X, J\,dH) = dH(X) \quad \text{with } X \in T_x\Gamma. \quad (1)$$

a) We defined the Poisson bracket by

$$\{F, H\} := \omega(X_H, X_F)$$

where
$$\begin{cases} X_H(x) := J\,dH(x) = \left.\frac{d}{dt}\right|_{t=0} \phi_t(x) \\ X_F(x) := J\,dF(x) = \left.\frac{d}{ds}\right|_{s=0} \psi_s(x) \end{cases}$$

We then have

$$\{F,H\} = \omega(X_H, X_F) = \omega(X_H, J\,dF) \overset{(1)}{=} dF(X_H) \overset{(2)}{=} \left.\frac{d}{dt}\right|_{t=0} \phi_t^* F .$$

Let use recall the definition of the differential dF along the vector v:

$$\left.dF(v)\right|_x := \left.\frac{d}{dt}\right|_{t=0} F(\gamma_t(x)) = \left.\frac{d}{dt}\right|_{t=0} \left.\gamma_t^* F\right|_x \quad (2)$$

where $\gamma_{t=0}(x) = x$

and $v = \left.\frac{d}{dt}\right|_{t=0} \gamma_t(x)$.

It follow directly that

$$\{F, H\} = 0 \iff \left.\frac{d}{dt}\right|_{t=0} \phi_t^* F$$

b) F is a conserved quantity for ϕ_t if $\phi_t^* F = F \quad \forall t \in I$.
From a) it directly follows that the latter holds when $\{F, H\} = 0$.
We want to show that if F is a conserved quantity for ϕ_t, then H is a conserved quantity for ψ_s, i.e. $\psi_s^* H = \psi_s$.

We have

$$\phi_t^* F = F \quad \forall t \quad \Leftrightarrow \quad \{F, H\} = 0 .$$

From what precedes we can write

$$\{F, H\} := \omega(X_H, X_F) = -\omega(X_F, X_H)$$

$$= -\omega(X_F, J\,dH)$$

$$= -dH(X_F)$$

$$=: -\left.\frac{d}{ds}\right|_{s=0} \psi_s^* H ,$$

then

$$\{F, H\} = 0 \quad \Leftrightarrow \quad \psi_s^* H = \psi_{s=0}^* H = H .$$