Üb 4.1) [We demonstrate the properties c) and f)]

Let M be a $2n$-dimentional symplectic manifold with the symplectic structure ω (non-degenerate closed differential 2-form: $\omega : T_xM \times T_xM \to \mathbb{R}$). Let the local coordinates $x \equiv (p,q) = (p_1, q^1, \dots, p_n, q^n)$ be Darboux coordinates, such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i = dx^T J\, dx \,, \qquad (1)$$

with $J = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}$.

We write $\tilde{x} \equiv (P, Q) = (P_1, Q^1, \dots, P_n, Q^n)$ the new coordinates under the canonical map $\varphi : M \to M : (p,q) \mapsto (P,Q)$.

Let be F and G, two Hamiltonian functions defined on the phase space M. The associated Hamiltonian vector fields are given by

$$X_F = J\, dF \quad \text{and} \quad X_G = J\, dG \,. \qquad (2)$$

We have defined the Poisson bracket of the two functions F and G as:

$$\{F, G\} := -\,\omega(X_F, X_G) \,. \qquad (3)$$

[We defined the mapping $J : T_x^*M \to T_xM : dF \mapsto J\,dF = X_F$, of a differential one-form to a vector field.]

Note that this definition does not involve local coordinates, then we expect the Poisson bracket to be invariant under a specific choice of (canonical) local coordinates. Nevertheless, let us show it explicitely.

e) Specifying the local coordinates, we write

$$\{F, G\}_{\tilde{x}=(P,Q)} = \omega\left(X_{G(P,Q)}, X_{F(P,Q)}\right)$$

$$= \omega\left(X_{G\varphi(p,q)}, X_{F\varphi(p,q)}\right)$$

$$= \omega\left(\varphi_* X_{G(p,q)}, \varphi_* X_{F(p,q)}\right)$$

$$= (\varphi^* \omega)\left(X_G, X_F\right)_{(p,q)}$$

φ is canonical $\longrightarrow$ $= \omega(X_G, X_F)_{(p,q)}$

$$=: \{F, G\}_{x=(p,q)} .$$

f) From the definition (3) and (2), we have

$$\{F, G\}_{x=(p,q)} = \omega\left(J\, dG, J\, dF\right)_{(p,q)}$$

$$= \omega\left(J\, dG_{(p,q)}, J\, dF_{(p,q)}\right)$$

$$= \omega\left(J \frac{\partial G}{\partial x}, J \frac{\partial F}{\partial x}\right)$$

Let us consider the simple case of $n=1$: $x=(p,q)$ and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
such that

$$\{F, G\}_x = \omega\left(\begin{pmatrix} -\frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix}, \begin{pmatrix} -\frac{\partial F}{\partial q} \\ \frac{\partial F}{\partial p} \end{pmatrix}\right) = \begin{vmatrix} -\frac{\partial G}{\partial q} & -\frac{\partial F}{\partial q} \\ \frac{\partial G}{\partial p} & \frac{\partial F}{\partial p} \end{vmatrix} = \frac{\partial G}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial G}{\partial q}\frac{\partial F}{\partial p}$$

Let us note that

$$\omega(\xi_1, \xi_2)_x = (\xi_1, J\xi_2)_x = \xi_1^T(x)\, J \;\; \xi_2(x)$$

$$\forall\, \xi_1, \xi_2 \in T_xM$$

We then have

$$\{F, G\}_x = \omega(J\,dG, J\,dF)_x$$

$$= (J\,dG_x)^T \; J\,(J\,dF_x)$$

$$= \left(\frac{\partial G}{\partial x}\right)^T \underbrace{(J^T J)}_{-\mathbb{1}} \; J\left(\frac{\partial F}{\partial x}\right)$$

$$= -\left(\frac{\partial G}{\partial x}\right)^T J\left(\frac{\partial F}{\partial x}\right)$$

$$= -\left(\frac{\partial G}{\partial p_1} \;\; \frac{\partial G}{\partial q^1} \cdots \frac{\partial G}{\partial p_n} \;\; \frac{\partial G}{\partial q^n}\right) \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial p_1} \\ \frac{\partial F}{\partial q_1} \\ \vdots \\ \frac{\partial F}{\partial p_n} \\ \frac{\partial F}{\partial q^n} \end{pmatrix}$$

$$= -\left(\frac{\partial G}{\partial p_1} \;\; \frac{\partial G}{\partial q^1} \cdots \frac{\partial G}{\partial p_n} \;\; \frac{\partial G}{\partial q^n}\right) \begin{pmatrix} -\frac{\partial F}{\partial q_1} \\ \frac{\partial F}{\partial p_1} \\ \vdots \\ -\frac{\partial F}{\partial q^n} \\ \frac{\partial F}{\partial p_n} \end{pmatrix} = \frac{\partial G}{\partial p_1}\frac{\partial F}{\partial q_1} - \frac{\partial G}{\partial q_1}\frac{\partial F}{\partial p^1} + \cdots + \frac{\partial G}{\partial p_n}\frac{\partial F}{\partial q^n} - \frac{\partial G}{\partial q_n}\frac{\partial F}{\partial p_n}$$

$$= \sum_{i=1}^{n} \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q^i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p_i}$$