Solid State Theory
Exercise 11

FS 11
Prof. M. Sigrist

Transport in metals

Exercise 11.1 Relaxation time approximation

In this exercise we will show that the so-called single-relaxation-time approximation,

$$\left(\frac{\partial f(\boldsymbol{k})}{\partial t}\right)_{\text{coll}} = -\int \frac{\mathrm{d}^d k'}{(2\pi)^d} W(\boldsymbol{k},\boldsymbol{k}')[f(\boldsymbol{k}) - f(\boldsymbol{k}')] \quad \longrightarrow \quad -\frac{f(\boldsymbol{k}) - f_0(\boldsymbol{k})}{\tau}, \tag{1}$$

is a true solution to the Boltzmann equation under certain conditions.
We consider a homogeneous two-dimensional metal with an isotropic Fermi surface ($\varepsilon_{\boldsymbol{k}} = \hbar^2 k^2/2m$) at zero temperature. The impurity scattering responsible for a finite resistivity is described by a delta potential in real space,

$$V_{\text{imp}}(\boldsymbol{r}) = V_0\,\delta(\boldsymbol{r}). \tag{2}$$

The system is subject to a homogeneous and time-independent electric field along the x-axis.

a) Show that the transition rates $W(\boldsymbol{k},\boldsymbol{k}')$ for the impurity potential (2) are constant in k-space.

b) Write down the static Boltzmann transport equation for this setup in the form

$$\text{“drift-term”} = \text{“collision-integral”} \tag{3}$$

and take advantage of the zero-temperature limit and the symmetries of the system to eliminate all but angular variables.

c) In a case with only angular dependence, it turns out to be useful to expand the drift term $\boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot (e\boldsymbol{E})$ and $\delta f = f - f_0$ in Fourier modes

$$\delta f = \sum_l f_l\, e^{il\varphi}, \qquad \boldsymbol{\nabla}_{\boldsymbol{k}} f \cdot (e\boldsymbol{E}) = \sum_l d_l\, e^{il\varphi}. \tag{4}$$

Rewrite the Boltzmann equation as a set of algebraic equations for the coefficients in the expansion (4)

$$d_m = \sum_n L_{m,n} f_n. \tag{5}$$

d) What are the eigenvalues of the so-called collision operator $L_{m,n}$ and what is their meaning? How can one interpret vanishing eigenvalues?

e) Find a solution to the equation (5) and compare δf to the single-relaxation-time approximation, equation (1).

Exercise 11.2 Penetration depth in a superconductor

We consider a superconductor with a normal-conducting component ρ_n and a superconducting component ρ_s, where $\rho = \rho_n + \rho_s$ is the total electron density. The conductivity of the system is given by the conductivity of the two components, $\sigma = \sigma_n + \sigma_s$, with

$$\sigma_n(\omega) = \rho_n \frac{e^2}{m} \frac{\tau}{1 - i\omega\tau} \quad \text{and} \quad \sigma_s(\omega) = i\rho_s \frac{e^2}{m(\omega + i0^+)}. \tag{6}$$

The density of the superconducting component depends on the temperature in the following way (Gorter-Casimir two-fluid model)

$$\rho_s(T) = \rho \left[1 - \left(\frac{T}{T_c} \right)^4 \right]. \tag{7}$$

a) Use the expression for the dielectric constant of a metal $\varepsilon(\omega) = 1 + (4\pi i/\omega)\sigma(\omega)$ in order to compute the penetration depth $\delta(\omega, T)$ in the limit $\omega\tau \ll 1$.

b) Plot the penetration depth $\delta(\omega, T)$ for small ω as a function of the frequency ω and temperature T. Discuss the limits $T \to T_c$ and $T \to 0$.

Exercise 11.3 Reflectivity of Simple Metals and Semiconductors

Use the expression for the Drude conductivity,

$$\sigma(\omega) = \frac{\omega_p^2}{4\pi} \frac{\tau}{1 - i\omega\tau}, \tag{8}$$

to obtain an expression for the reflectivity $R(\omega)$ of a simple metal or semiconductor using the connection between $\sigma(\omega)$ and $R(\omega)$ given in section 6.2.2. of the lecture notes. To take into account the effect of the bound (or core) electrons, use as a phenomenological ansatz for the dielectric function

$$\epsilon(\omega) = \epsilon_\infty + \epsilon_{\text{Drude}}(\omega) - 1. \tag{9}$$

Here ϵ_∞ is assumed to be constant in the frequency range of interest, related to the fact that the energy scale for exciting core electrons is much higher than the typical energy scales for the itinerant electrons. Plot the reflectivity for the cases $\epsilon_\infty = 1$ and $\epsilon_\infty = 20$ and $\tau\omega_p = \infty, 40, 2$!
Usually, ϵ_∞ is much larger in semiconductors than in metals. Can you think about possible explanations for this behavior?

Office hour:
Monday, May 16th, 2011 - 9:00 to 11:00 am
HIT K 11.3
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