Resource inequalities

Quantum Information Theory

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Some types of resources

Alice

- $\overset{n}{\rightsquigarrow}$ perfect quantum channel
  (Alice sends *n* qubits to Bob)
- $\overset{n}{\rightarrow}$ perfect classical channel
  (Alice sends *n* bits to Bob)
- $\overset{n}{\nrightarrow}$ noisy classical channel
- $\underset{n}{\sim}$ shared entanglement, or *ebits*
  (Alice and Bob share *n* Bell pairs)
- $\overline{n}$ shared bits

Bob

Resource inequalities

Definition: $X \geq Y$ means "we can obtain Y using X".
Formally, there exists a protocol to simulate resources Y using only resources X and local operations.

Examples

$\overset{1}{\rightsquigarrow} \geq \overset{\sim\sim}{1}$ (entanglement distribution)

- Alice prepares an entangled pair, $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, locally.
- She sends one of the qubits to Bob through the quantum channel.

$\overset{n}{\nrightarrow} \geq \overset{n\ I(A:B)}{\longrightarrow}$, in the limit $n \to \infty$.

- Channel coding for iid channels (p. 21 of the script).

Superdense coding

$$\overset{1}{\rightsquigarrow} \; \underset{1}{\sim} \; \geq \; \overset{2}{\rightarrow}$$

Goal:
Alice wants to send two classical bits, i and j, to Bob.
They share one Bell state. She can also send him one qubit.

Protocol:

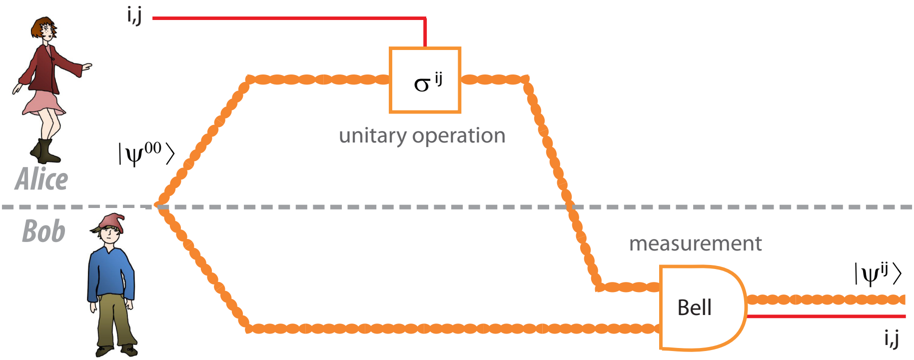
1. Alice applies a local unitary operation, σ^{ij}, on her half of the entangled state.
   Here, σ^{ij} are the Pauli matrices: $\sigma^{00} = \mathbb{1}, \sigma^{01} = \sigma^{x}$, etc.

i, j	Global operation	Resulting state
00	$\mathbb{1}_A \otimes \mathbb{1}_B \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}} =: \lvert\psi^{00}\rangle$
01	$\sigma^x_A \otimes \mathbb{1}_B \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 00\rangle - \lvert 11\rangle}{\sqrt{2}} =: \lvert\psi^{01}\rangle$
10	$\sigma^y_A \otimes \mathbb{1}_B \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 01\rangle + \lvert 10\rangle}{\sqrt{2}} =: \lvert\psi^{10}\rangle$
11	$\sigma^z_A \otimes \mathbb{1}_B \frac{\lvert 00\rangle + \lvert 11\rangle}{\sqrt{2}}$	$\frac{\lvert 01\rangle - \lvert 10\rangle}{\sqrt{2}} =: \lvert\psi^{11}\rangle$

The states $\lvert\psi^{ij}\rangle$ form a basis for two qubits: the Bell basis.

Superdense coding

2. Alice sends her qubit to Bob.
3. Bob measures the two qubits in the Bell basis.
   Outcome of his measurement: i, j.



Teleportation

$$\overset{2}{\rightarrow} \; \underset{1}{\sim} \; \geq \; \overset{1}{\rightsquigarrow}$$

Goal:
Alice wants to communicate the state of one qubit, S, to Bob. They share one Bell state. She can also send him two classical bits.

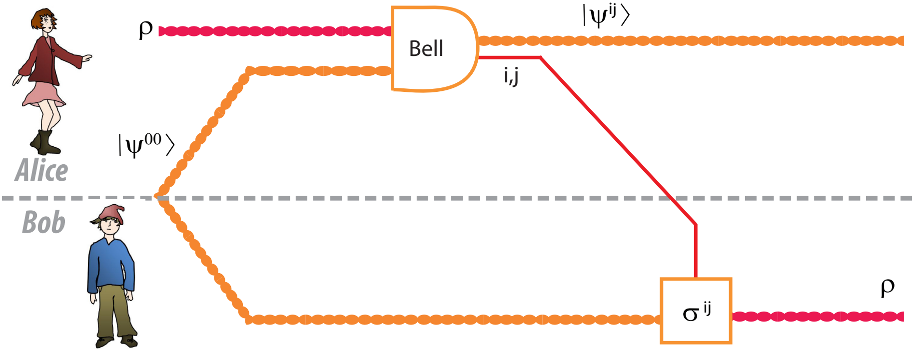
Consider that S is in a pure state, $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$. (general case in ex. 11.1)
Global state: $|\phi\rangle_S \otimes |\psi^{00}\rangle_{AB}$.

Protocol:
1. Alice measures *S* and *A* (her half of the entangled state) in the Bell basis.

Alice's outcome	Global projector	Resulting global state
$00: \; |\psi^{00}\rangle_{SA}$	$|\psi^{00}\rangle\langle\psi^{00}|_{SA} \otimes \mathbb{1}_B$	$|\psi^{00}\rangle_{SA} \otimes (\alpha|0\rangle + \beta|1\rangle)_B$
$01: \; |\psi^{01}\rangle_{SA}$	$|\psi^{01}\rangle\langle\psi^{01}|_{SA} \otimes \mathbb{1}_B$	$|\psi^{01}\rangle_{SA} \otimes (\alpha|0\rangle - \beta|1\rangle)_B$
$10: \; |\psi^{10}\rangle_{SA}$	$|\psi^{10}\rangle\langle\psi^{10}|_{SA} \otimes \mathbb{1}_B$	$|\psi^{10}\rangle_{SA} \otimes (\beta|0\rangle + \alpha|1\rangle)_B$
$11: \; |\psi^{11}\rangle_{SA}$	$|\psi^{11}\rangle\langle\psi^{11}|_{SA} \otimes \mathbb{1}_B$	$|\psi^{11}\rangle_{SA} \otimes (\beta|0\rangle - \alpha|1\rangle)_B$

Teleportation

2. Alice sends the classical bits that describe her outcome, i, j, to Bob.
3. Bob applies σ^{ij} on his qubit.
   Resulting state: $|\phi\rangle$.



Teleportation preserves entanglement between ρ and the rest of the universe.

More RI: teleportation and entanglement

Suppose that Alice can send unlimited classical communication to Bob. How much entanglement do they need in order to transmit one qubit?

$$\overset{\infty}{\longrightarrow} + \underset{n}{\sim\!\sim} \;\geq\; \overset{m}{\rightsquigarrow}$$

Can we have $m > n$?

No! Proof:

We are going to define a monotone, E, such that:

1. $E\left(\underset{n}{\sim\!\sim}\right) = n$;
2. $E\left(\overset{m}{\rightsquigarrow}\right) = m$;
3. E can only decrease under the operations allowed by this RI: arbitrary classical communication and local operations.

This will give us $E\left(\overset{m}{\rightsquigarrow}\right) \leq E\left(\underset{n}{\sim\!\sim}\right)$, or $m \leq n$.

More RI: teleportation and entanglement

Squashed entanglement

$$E(A:B) := \frac{1}{2}\min_R I(A:B|R)$$

Before: $A \underset{n}{\rightsquigarrow} B$

ρ_{AB}: n maximally entangled pairs of qubits $\Rightarrow$ pure state $\Rightarrow$ $\rho_{ABR} = \rho_{AB} \otimes \rho_R$

$$I(A:B|R) = I(A:B) = 2n, \forall R \quad \Rightarrow \quad E(A:B) = n$$

After: $A \overset{m}{\rightsquigarrow} B$

$\overset{m}{\rightsquigarrow} \geq \tilde{m}$ Alice prepares m ebits and sends half of each to Bob.

$$E(A:B) = m$$

More RI: teleportation and entanglement

$$2E(A:B) = \min_R I(A:B|R)$$

$E(A:B)$ can only decrease under:

Local operations

Because $I(A:B|R)$ cannot increase under local operations.

Classical communication

Alice will send classical system C to Bob (e.g. a bit string).
We want to compare $E(AC:B)$ and $E(A:BC)$.

$$\begin{aligned}
\exists R : 2E(AC:B) &= I(B:AC|R) \quad \text{(the same as } I(AC:B|R)\text{)} \\
&= H(B|R) - H(B|ACR) \\
&\geq H(B|RC) - H(B|ARC) \quad \text{(strong subadditivity)} \\
&= I(B:A|RC) \\
&= I(BC:A|RC) \quad RC =: R' \\
&\geq \min_{R'} I(BC:A|R') = 2E(A:BC)
\end{aligned}$$

More RI: teleportation and classical communication

Suppose that Alice and Bob share unlimited entanglement.
Can Alice send Bob n qubits by sending him less than $2n$ classical bits?

$$\begin{array}{c} \overset{m}{\rightarrow} \\ \overset{\sim\sim}{\infty} \end{array} \geq \begin{array}{c} \overset{n}{\rightsquigarrow} \\ \overset{\sim\sim}{\infty} \end{array}$$

Can we have $m < 2n$?

No! Proof:

Concatenate teleportation and superdense coding
(with unlimited entanglement).

$$\begin{array}{c} \overset{m}{\rightarrow} \\ \overset{\sim\sim}{\infty} \end{array} \geq \begin{array}{c} \overset{n}{\rightsquigarrow} \\ \overset{\sim\sim}{\infty} \end{array} \qquad \begin{array}{c} \overset{n'}{\rightsquigarrow} \\ \overset{\sim\sim}{\infty} \end{array} \geq \begin{array}{c} \overset{m'}{\rightarrow} \\ \overset{\sim\sim}{\infty} \end{array}$$

Fix $n = n'$:

$$\begin{array}{c} \overset{m}{\rightarrow} \\ \overset{\sim\sim}{\infty} \end{array} \geq \begin{array}{c} \overset{n}{\rightsquigarrow} \\ \overset{\sim\sim}{\infty} \end{array} \geq \begin{array}{c} \overset{m'}{\rightarrow} \\ \overset{\sim\sim}{\infty} \end{array}$$

More RI: teleportation and classical communication

$$\underset{\overset{\approx}{\infty}}{\overset{m}{\rightarrow}} \geq \underset{\overset{\approx}{\infty}}{\overset{n}{\rightsquigarrow}} \geq \underset{\overset{\approx}{\infty}}{\overset{m'}{\rightarrow}}$$

Assume that for $\underset{\overset{\approx}{\infty}}{\overset{m}{\rightarrow}} \geq \underset{\overset{\approx}{\infty}}{\overset{m'}{\rightarrow}}$ we need $m \geq m'$. (ex. 11.3)

From superdense coding we know that we can have $\underset{\overset{\approx}{\infty}}{\overset{n}{\rightsquigarrow}} \geq \underset{\overset{\approx}{\infty}}{\overset{2n}{\rightarrow}}$.

Therefore we have $\underset{\overset{\approx}{\infty}}{\overset{m}{\rightarrow}} \geq \underset{\overset{\approx}{\infty}}{\overset{n}{\rightsquigarrow}} \geq \underset{\overset{\approx}{\infty}}{\overset{2n}{\rightarrow}}$, and $m \geq m' = 2n$.

More RI: “hyperdense coding”?

Suppose that Alice and Bob share unlimited entanglement.
Is it possible to send more than two bits, by sending only one qubit?

$$\begin{matrix}\overset{n}{\rightsquigarrow}\\ \overset{\sim\sim}{\infty}\end{matrix} \geq \begin{matrix}\overset{m}{\rightarrow}\\ \overset{\sim\sim}{\infty}\end{matrix}$$

Can we have $m > 2n$?

No! Proof:

Concatenate superdense coding and teleportation
(with unlimited entanglement).

$$\begin{matrix}\overset{n}{\rightsquigarrow}\\ \overset{\sim\sim}{\infty}\end{matrix} \geq \begin{matrix}\overset{m}{\rightarrow}\\ \overset{\sim\sim}{\infty}\end{matrix} \qquad \begin{matrix}\overset{2n'}{\rightarrow}\\ \overset{\sim\sim}{\infty}\end{matrix} \geq \begin{matrix}\overset{n'}{\rightsquigarrow}\\ \overset{\sim\sim}{\infty}\end{matrix}$$

Fix $m = 2n'$:

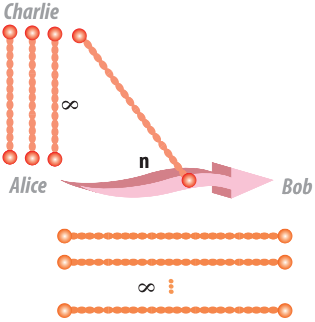
$$\begin{matrix}\overset{n}{\rightsquigarrow}\\ \overset{\sim\sim}{\infty}\end{matrix} \geq \begin{matrix}\overset{2n'}{\rightarrow}\\ \overset{\sim\sim}{\infty}\end{matrix} \geq \begin{matrix}\overset{n'}{\rightsquigarrow}\\ \overset{\sim\sim}{\infty}\end{matrix}$$

More RI: “hyperdense coding”?

$$\begin{matrix}\overset{n}{\rightsquigarrow}\\ \underset{\infty}{\approx}\end{matrix} \geq \begin{matrix}\overset{2n'}{\rightarrow}\\ \underset{\infty}{\approx}\end{matrix} \geq \begin{matrix}\overset{n'}{\rightsquigarrow}\\ \underset{\infty}{\approx}\end{matrix}$$

We just have to show that in order to have $\begin{matrix}\overset{n}{\rightsquigarrow}\\ \underset{\infty}{\approx}\end{matrix} \geq \begin{matrix}\overset{n'}{\rightsquigarrow}\\ \underset{\infty}{\approx}\end{matrix}$ we need $n \geq n'$.

Again, we are going to define a monotone. Consider the following setting:



1. Alice shares ∞ e-bits with Bob and ∞ e-bits with a third player, Charlie.
2. Alice has an n-qubit quantum channel to Bob.
3. Other than these resources, only local operations are allowed.
4. The goal is to maximize $\Delta I(B:C)$, the difference between initial and final mutual information between Bob and Charlie.

More RI: "hyperdense coding"?

Why this example?

Because it simulates a quantum channel from Charlie to Bob:
$\Delta I(B:C) = 2n'$.

In general, Bob starts with system B_0, receives a quantum system Q from Alice (of at most n qubits) and then applies a local TPCPM, so that $B_F = \mathcal{E}(B_0 Q)$ (sorry for the abuse of notation).

We have

$$\begin{aligned} I(B_F:C) &= I(\mathcal{E}(B_0 Q):C) \\ &\le I(B_0 Q:C) && \text{(strong subadditivity)} \\ &= I(B_0:C) + I(Q:C|B_0) && \text{(chain rule; check for yourself)} \\ &\le I(B_0:C) + 2n && \text{(because } \log_2 |Q| \le n\text{)} \\ \Delta I(B:C) &\le 2n \\ 2n' &\le 2n. \end{aligned}$$