Quantenmechanik II
Serie 4

Aufgabe 4.1 Pauli Paramagnetism

The simplest model that can be used to understand the paramagnetic properties of conduction electrons in a metallic solid is the *free electron model*. Put in a few words, this approximation basically assumes that the conduction electrons in a metal are completely ionized from their parent atoms, and behave like a gas of free electrons wandering around in the solid.

Use the free electron approximation to calculate the paramagnetic susceptibility of a metal. To do so, start by considering a free electron gas contained in a cubic box of length L, and follow these steps:

(a) Show that imposing periodic boundary conditions the energy eigenvalues of the single particle states are

$$E_{\mathbf{k}} = \frac{\hbar^2 |\mathbf{k}|^2}{2m}, \tag{1}$$

where

$$\mathbf{k} = \frac{2\pi}{L}\left(n_x \mathbf{e}_x + n_y \mathbf{e}_y + n_z \mathbf{e}_z\right) \qquad n_x, n_y, n_z \in \mathbb{Z} \tag{2}$$

(b) In a macroscopic solid L is very large, so it can be assumed that the spectrum of the allowed $\mathbf{k}$ vectors is effectively continuous, with each single particle state occupying a volume $(2\pi/L)^3$ in momentum space.

Show that under such an approximation the number of single particle states with an energy eigenvalue lower or equal than E is

$$\frac{L^3}{3\pi^2}\left(\frac{2mE}{\hbar^2}\right)^{3/2} \tag{3}$$

(Remember that for each allowed $\mathbf{k}$ vector there are two single particle states).

(c) Use the result obtained in (b) to show that the number of single particle states with an energy eigenvalue between E and $E + dE$ is

$$D(E)\mathrm{d}E \tag{4}$$

where

$$D(E) = \frac{L^3}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} \tag{5}$$

(d) By Pauli exclusion principle only two electrons (one with spin up, and one with spin down) are allowed in a state with the same wavevector $\mathbf{k}$. Therefore, in the lowest energy state of the gas, the electrons fill up all the energy levels till a certain energy E_F, occupying all the allowed $\mathbf{k}$ vectors inside a sphere of radius $|\mathbf{k}_F|$ in momentum space. Equivalently it can be said that if the number of electrons in the gas is N, then E_F is the energy level below which there are N single particle states or $N/2$ allowed wavevectors.

Find the relation between N and E_F (if you solved part (b) this should be trivial), and show that

$$\int_0^{E_F} D(E)dE = N \tag{6}$$

(e) So far it could be fairly assumed that the number of spin up and spin down electrons are the same

$$N_+ = N_- = \frac{N}{2} = \frac{1}{2}\int_0^{E_F} D(E)dE. \tag{7}$$

However, if a magnetic field $\mathbf{B} = B\mathbf{e}_z$ is turned on, the energy of the spin down electrons is lowered by $-\mu_B B$ while the energy of the electrons with spin up is increased by $\mu_B B$. Knowing that the magnetization density is given by

$$M = -\mu_B(N_+ - N_-), \tag{8}$$

and assuming that the magnetic field strength is weak enough so that the change in the value of E_F calculated in (d) can be neglected, calculate the paramagnetic susceptibility

$$\chi = \frac{M}{B}. \tag{9}$$