

# Path integral formalism

(script sections 2 and 3)

## Contents

A	Motivation	1
B	Path integral in QM (script section 2.1)	2
C	Path integral for scalar QFT (script section 3)	3
D	Greens functions (script section 2.2 and 2.4 but for fields)	3
E	Generating functional $Z[J]$ (script sections 2.2 and 2.3 but for fields)	3
F	Free propagator from $Z[J]$ (script section 3.1)	3
G	Euclidean formulation and statistical mechanics	4

Script section numbers refer to lecture notes of Prof. Babis Anastasiou.

## A Motivation

*Advantages* of the path integral formalism wrt. “canonical quantization”:

- compact notation
- covers non-perturbative features
- natural interface to lattice gauge theory
- encodes  $n$ -point functions for all  $n$  (effect. potential)
- explicit conservation symmetries (incl. Lorentz symmetry)
- easy derivation of Feynman rules
- easy quantization of non-abelian QFT (Fadeev-Popov method)

*Literature (formal level)*: any modern QFT text book, e.g.:

- Peskin, Schroeder: An introduction to quantum field theory
- Weinberg: Quantum theory of fields I
- Ryder: Quantum field theory

*Literature (in-depth, related fields)*:

- Roepstorff: Pfadintegrale in der Quantenphysik
- Zinn-Justin: Quantum field theory and critical phenomena

## B Path integral in QM

(script section 2.1)

Assume

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (1)$$

Show:

$$\langle x_f | e^{-i\hat{H}(t_f-t_i)} | x_i \rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}x \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(x(t))\right) \quad (2)$$

R.h.s. is the Feynman path integral of quantum mechanics. “Integration over all paths”.

Key points:

- transition from operators to functions (distrib.)
- limit  $\delta t \rightarrow 0$  gives linearization in  $\delta t$

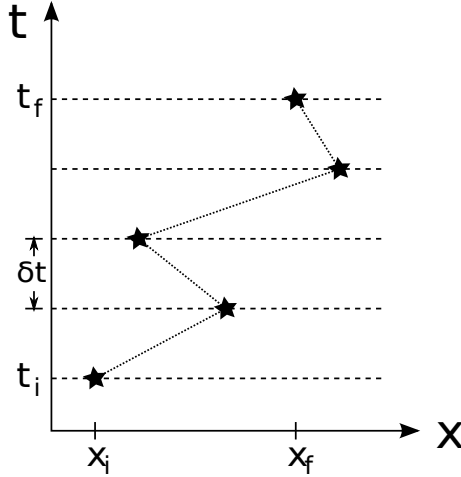


Figure 1: Supporting points for some path with  $x(t_i) = x_i$  and  $x(t_f) = x_f$ .

Note: it is straight-forward to generalize this to  $N$  dimensions:

$$\langle \vec{x}_f | e^{-i\hat{H}(t_f-t_i)} | \vec{x}_i \rangle = \frac{1}{\mathcal{N}} \int \mathcal{D}\vec{x} \exp\left(\frac{i}{\hbar} \int_{t_i}^{t_f} dt L(\vec{x}(t))\right) \quad (3)$$

where

$$\vec{x} \equiv (x_1, \dots, x_N), \quad \mathcal{D}\vec{x} \equiv \prod_{j=1}^N \mathcal{D}x_j. \quad (4)$$

Note: generalization to Hamiltonians with more complicated  $\hat{p}_j$  dependence

$$\langle \vec{q}_f | e^{-i\hat{H}(t_f-t_i)} | \vec{q}_i \rangle = \int \prod_{i=1}^N \mathcal{D}q_i \prod_{j=1}^N \mathcal{D}p_j \exp\left(\frac{i}{\hbar} \int_{t_1}^{t_2} dt \left(\sum_{k=1}^N p_k \dot{q}_k - H(p, q)\right)\right) \quad (5)$$

with integration over all paths  $\vec{p}(t)$  and all paths  $\vec{q}(t)$  which match boundary conditions  $\vec{q}(t_i) = \vec{q}_i$  and  $\vec{q}(t_f) = \vec{q}_f$  at the endpoints. Must take care of  $\hat{p}_j$  and  $\hat{q}_j$  ordering.

## C Path integral for scalar QFT

(script section 3)

Consider scalar QFT as quantum mechanical system with “infinitely many oscillators” (and use high energy units with  $\hbar = 1$ ):

$$[\hat{x}_i, \hat{p}_j] = i\delta_{ij} \quad (6)$$

$$\rightarrow [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad (7)$$

Postulate: in analogy to the QM case assume that in QFT holds

$$\langle \phi | e^{-i\hat{H}(t_f - t_i)} | \phi' \rangle = \mathcal{N} \int \mathcal{D}\phi \exp\left(i \int_{t_i}^{t_f} dt L\right) \quad (8)$$

where the r.h.s. is the QFT path integral,  $L = \int d^3\vec{x} \mathcal{L}$  and integration is over all field configurations with  $\phi(\vec{x}, t_i) = \phi_i(\vec{x})$  and  $\phi(\vec{x}, t_f) = \phi_f(\vec{x})$ . Show in the following this makes sense.

## D Greens functions

(script section 2.2 and 2.4 but for fields)

Show:

$$\langle \phi | e^{-i\hat{H}t_f} T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) e^{i\hat{H}t_i} | \phi' \rangle = \int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp\left(i \int_{t_i}^{t_f} dt L\right) \quad (9)$$

Show:

$$\langle 0 | T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | 0 \rangle = \frac{\int \mathcal{D}\phi \phi(x_1) \cdots \phi(x_n) \exp(i \int d^4\vec{x} \mathcal{L})}{\int \mathcal{D}\phi \exp(i \int d^4\vec{x} \mathcal{L})} \quad (10)$$

using a “damping term”  $(1/2)i\epsilon\phi^2$  in  $\mathcal{L}$ .

## E Generating functional $Z[J]$

(script sections 2.2 and 2.3 but for fields)

Show:

$$\langle 0 | T(\hat{\phi}_H(x_1) \cdots \hat{\phi}_H(x_n)) | 0 \rangle = \frac{1}{i^n Z[0]} \frac{\delta^{(n)}}{\delta J(x_1) \cdots \delta J(x_n)} Z[J] \Big|_{J=0} \quad (11)$$

## F Free propagator from $Z[J]$

(script section 3.1)

Assume: free massive scalar theory

Show:

$$Z[J] = Z[0] \exp\left(-\frac{1}{2} \int d^4x d^4y J(x) D_F(x-y) J(y)\right) \quad (12)$$

Then we get immediately:

$$\langle 0 | T(\hat{\phi}_H(x_1) \hat{\phi}_H(x_n)) | 0 \rangle = D_F(x_1 - x_n) \quad (13)$$

$$= \int \frac{d^4x}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \frac{i}{k^2 - m^2 + i\epsilon} \quad (14)$$

## G Euclidean formulation and statistical mechanics

Perform Wick rotation clockwise in complex  $t$ -plane, with

$$t \equiv -ix_4, \quad x_E = (x_1, x_2, x_3, x_4), \quad x^2 = -x_E^2, \quad (15)$$

With

$$\mathcal{L}_E = \frac{1}{2} \partial_\mu^E \phi \partial_\mu^E \phi + \frac{1}{2} m^2 \phi^2 \quad (16)$$

the path integral is:

$$Z[J] = \int \mathcal{D}\phi \exp \left( -\frac{1}{\hbar} \int d^4 x_E (\mathcal{L}_E - J\phi) \right) \quad (17)$$

Similar to partition function in statistical mechanics:

$$Z = \sum_n \exp \left( -\frac{1}{kT} E_n \right). \quad (18)$$