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Quantum Field Theory II
Exercise sheet 4

FS 10
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Exercise 4.1 Perturbative expansion of the four-point function within the $\lambda\phi^4$ theory

Consider a real scalar field ϕ of mass m with a ϕ^4 self-interaction proportional to $\lambda \ll 1$

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I \\ \mathcal{L}_0 &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\left(m^2 - i\epsilon\right)\phi^2 \\ \mathcal{L}_I &= -\frac{1}{4!}\lambda\phi^4\end{aligned}$$

According to the lecture, the generating functional is defined as

$$Z[J] = \frac{\exp\left[i\int d^4x\ \mathcal{L}_I\left(i\frac{\delta}{\delta J(x)}\right)\right] Z_0[J]}{\exp\left[i\int d^4x\ \mathcal{L}_I\left(i\frac{\delta}{\delta J(x)}\right)\right] Z_0[J]\Big|_{J=0}}$$

where $Z_0[J]$ is the generating functional for the free field

$$Z_0[J] = Z_0[0]\exp\left[-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right]$$

and $\mathcal{L}_I\left(i\frac{\delta}{\delta J(x)}\right)$ accounts for replacing $\phi(x)$ in the interaction Lagrangian by the functional derivative, in this case

$$\mathcal{L}_I\left(i\frac{\delta}{\delta J(x)}\right) = -\frac{\lambda}{4!}\frac{\delta^4}{\delta J(x)^4}$$

a) Compute to order λ the four-point function

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \frac{1}{i^4}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}Z[J]\Big|_{J=0}$$

and draw the corresponding diagrams.

b) Compute to order λ the *connected* four-point function

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle_{\text{connected}} = \frac{i}{i^4}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}W[J]\Big|_{J=0}$$

where

$$W[J] = -i\log Z[J] \tag{1}$$

and verify that the corresponding diagrams are indeed connected.

c) *[optional]* Compute the connected four-point function to order λ^2.

Exercise 4.2 Grassmann numbers

Let $\{c_i\}$ be Grassmann numbers, i.e. $c_i c_j = -c_j c_i$.
Show that

$$\left\{c_i, \frac{\partial}{\partial c_j}\right\} = \delta_{ij} \qquad \text{and} \qquad \left\{\frac{\partial}{\partial c_i}, \frac{\partial}{\partial c_j}\right\} = 0$$

where $\frac{\partial}{\partial c_i}$ is the *left*-derivative, namely

$$\frac{\partial}{\partial c_i}(c_i c_j) = c_j \qquad \text{but} \qquad \frac{\partial}{\partial c_i}(c_j c_i) = \frac{\partial}{\partial c_i}(-c_i c_j) = -c_j \qquad (\text{for } i \neq j)$$

Hint: show that any function f of c_i, c_j and other variables represented by α can be expanded as

$$f(c_i, c_j, \alpha) = f_1(\alpha) + c_i f_2(\alpha) + c_j f_3(\alpha) + c_i c_j f_4(\alpha)$$

and apply the anticommutators above on this function.

Exercise 4.3 Gaussian integrals with fermions

Using the definition of Grassmann integration

$$\int dc\ 1 = 0 \qquad \int dc\ c = 1$$

show that for two N-dimensional vectors $x = (x_1, \dots, x_N)^T$ and $y = (y_1, \dots, y_N)^T$ of Grassmann variables, and for a $N \times N$ matrix A of 'normal' (commuting) numbers, one has

$$\int dx_1 \dots dx_N dy_1 \dots dy_N\ e^{-x^T A y} = \det A$$