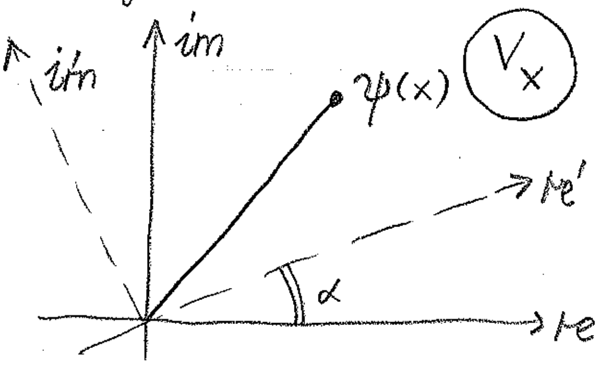
Gauge theories and geometry

As is well known, the principles of general relativity imply that the motion of particles in a gravitational field can be described by geometric tools. The point here is that also gauge fields can be viewed in this way.

Consider a field $\psi(x)$, $x \in M$ (space time, base manifold). The values of $\psi(x)$ lie typically in a vector space V over the complex numbers. For instance, a charged scalar field is in $\mathbb{R}^2$ (complex numbers).
Implicit in the value of $\psi(x)$ is a choice of a reference frame. Not only for the space-time axes at x, but also for V. In the above example this is the direction of the real and imaginary axes. Depending on the choice, the value



of ψ varies. If $\psi(x) = \rho e^{i\varphi}$ in system $\{re, im\}$, it is
$\psi(x) = e^{-i\alpha}\rho e^{i\varphi} = \rho e^{i(\varphi-\alpha)}$.
In the usual treatment, the freedom of considering different "internal" coordinate spaces is not taken into account. However this example shows that a "correct" description should include this.
Let us denote the space of all reference frames at x by P_x. In thes case above this are all values of α, with $\alpha + 2\pi n$ identified with α; thus P_x is a circle.

Under a gauge transformation, $\psi(x) \to e^{i\gamma(x)}\psi(x)$. This corresponds to a rotation of the frame in the "negative direction (clockwise). We see that the "space of the group", the complex numbers with unit length, is the same as the space P_x.

Two reference frames in P_x are related by an element of the group $U(1)$; if $p \in P_x$, then pg (with $g \in U(1)$) is the transformed frame. Writing $g = e^{i\gamma}$, we see from above

System $p = \{re, im\}$

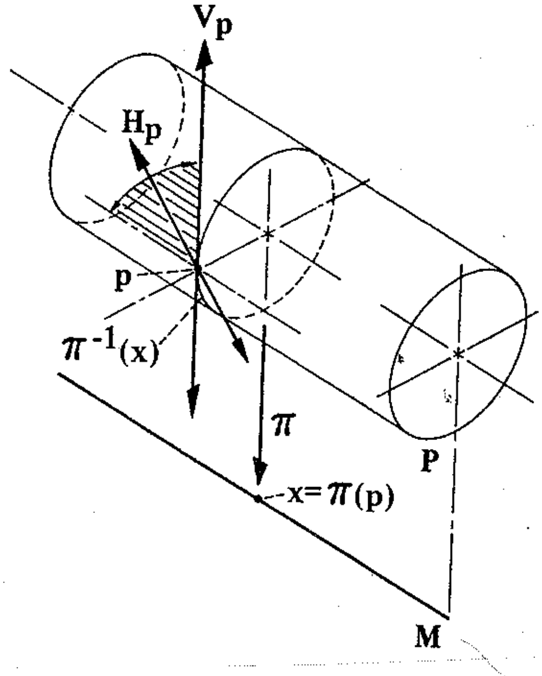
System $p' = \{re', im'\} = p e^{i\gamma} = pg$

$$\psi/_p = \rho e^{i\varphi} \qquad \psi/_{p'} = \rho e^{-i\gamma} \Leftrightarrow \qquad \psi/_{p'} = g^{-1}\psi/_p \ * \qquad (1)$$

Thus, the field ψ is not just a function of x, but also of $p \in P_x$; we may write $\psi(x,p)$. Only if we fix p for each x by some prescription, we have $\psi(x)$ (gauge choice, below).

We generalize the above to a group G with elements g.

A smooth union of all P_x (x goes over M) is called a principal fibre bundle with group G. P_x is the fiber above x. P is called the total space. There must be a mapping $\pi: P - M$ which associates to P the point x (via P_x). If $p \in P_x$, $g \to pg \ \forall g \in G$ gives a topological equivalent of G with P_x, but $P \neq M \times G$ in general



* $\psi/_p$ is the field relative to p.

if there is a twist in P. If $P = M \times G$ the PFB is "trivial". An example of a non trivial PFB is the Möbius-band, or S^3 is a $U(1)$ bundle over S^2. Trivial are the torus or a sausage. If $U \subset M$, then a function $\sigma_u : U \to P$ is called a gauge, if $\sigma_u(x) \in P_x$. This associates a $p \in P_x$ to x; and then we can consider the function $\psi(x,p)$ as function of x via $\psi(x,p) = \psi(x, \sigma_u(x))$. In this way we get the usual

notion of a function of x alone; it implies that the frame in P_x is fixed and thus "the "gauge", in which

which one works.
If in $W \in M$ we chose another gauge, $\sigma_W(x)$, then we can write $\sigma_W(x) = \sigma_u(x) g_{uw}(x)$ where $x \in U \cap W$ and $g_{uw}(x) : U \cap W \to G$. Then if

$$\psi_W(x,p) = \psi(x, \sigma_W(x))$$
$$\psi_u(x,p) = \psi(x, \sigma_u(x))$$

then $\psi_W(x,p) = \psi(x, \sigma_u(x) \cdot g_{uw}(x)) = g_{uw}^{-1} \psi(x, \sigma_u(x))$.
using (1) above. Thus we clearly see that σ_u is indeed a choice of gauge in the usual sense.
(In math language this is called local trivialization).

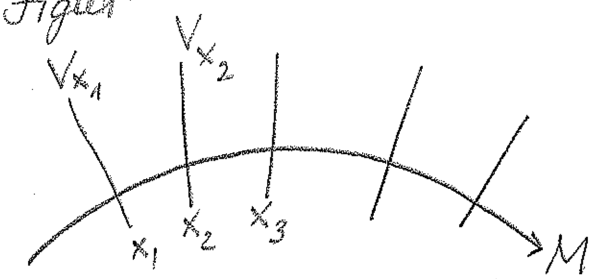
Comment: A union of spaces P attached at different points of the manifold M is a bundle. There must be a mapping $\pi : P \to M$. π essentially selects the fiber the base point x of the fiber P_x.
A common bundle is the union of tangent spaces $T_x M$, TM.

Covariant derivatives and connections

Die physik. Gleichungen sind i. A. Differentialgleichungen. Dazu müssen Quotienten der Art $d\psi/dx$ gebildet werden:

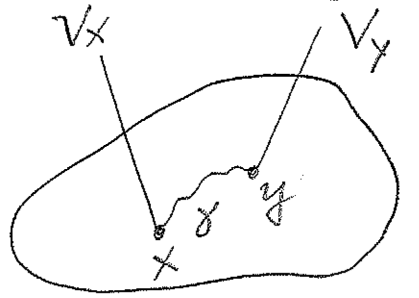
$$\frac{d\psi}{dx} = \frac{\psi(x+dx) - \psi(x)}{dx} \qquad \psi(x) \in V_x$$

Die rechte Seite is aber nicht wohl definiert, weil $\psi(x)$ und $\psi(x+dx)$ in verschiedenen Räumen sind; siehe Figur



$$V_{x_1} \neq V_{x_2}$$

We behachten speziell PFB und einen Weg γ und eine Funktion $\psi(x): x \to \psi(x) \in V_x$ ("ein "Schnitt" des Faserbündels $V = \bigcup_x V_x$). Da V_x un V_y Vektorräume sind, gibt es eine Abbildung



$$U(\gamma): V_x \longrightarrow V_y$$

wobei U von γ abhängen wird. Im Falle von PFB sind die $U(\gamma)$ Darstellungen von G, $V_x, V_y, ..$ die Darstellungsräume. Wir bilden entsprechend die "effective Differenz"

$$\text{"}\frac{\psi(y) - \psi(x)}{|y-x|}\text{"} = \frac{\psi(y) - U(\gamma)\psi(x)}{|y-x|} \qquad (D)$$

Diese ist sinnvoll, weil $U(\gamma)\psi(x)$, wie $\psi(y)$, in V_y liegt. Die Idee, die dieser Definition zugrunde liegt ist, dass eine reine Eichtransformation keine wirkliche Änderung bewirkt.

We now let the path be

$$y = x + \delta X, \quad X \in T_x M, \quad 0 \leq \delta \leq \varepsilon$$

(X is a vector in T_x; $X = \sum X^i \frac{\partial}{\partial x^i}$). ε is small, and we set $|x - y| = \varepsilon$. As mentioned $TM = \bigcup_{x \in M} T_x M$ is the tangent bundle.

Similarly we write

$$U(\gamma) = 1 + \varepsilon Z$$

where Z an element of the Lie-Algebra. More precisely if U is in the representation π of G, then Z is in the representation $d\pi$ of $\mathfrak{g}$, the Lie-Algebra. Z depends on X: It is a function on TM with values in $d\pi(\mathfrak{g})$. This implies that Z is in the dual space to TM,

$$T^*M \quad \text{(cotangent bundle)}$$

$T^*M = \bigcup_x T_x^* M$, T_x^* dual space of $T_x M$. The elements of T^*M are the 1-forms ω. In coordinates one often sets $\omega = \sum \omega^i dx^i$, where the dx^i form the basis of $T_x^* M$. If X is a general element of $T_x M$, then $\omega : X \to f$ ($f \in \mathbb{R}$, in $d\pi(\mathfrak{g}), \ldots$) with $f = (\omega, X) = \sum_{i,j} \omega^i X^j \underbrace{(dx_i, \frac{\partial}{\partial x^j})}_{\delta^i_j \leftarrow \text{ by assumption for } dx^i} = \sum \omega^i X^i$.

We therefore write

$$Z = Z(X) = \omega(X)$$

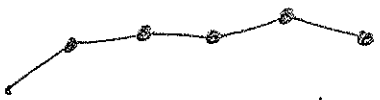
ω: 1-form with values in $d\pi(\mathfrak{g})$. We now set

$$\mathcal{U}(\gamma) = P \exp \int_\gamma \omega(X)$$

where P is the so called "path ordering" prescription.

To see this consider a finite path $\gamma : [0,t] \to M$, $\gamma(0) = x$, $\gamma(t) = y$ and divide it into small pieces. Then

$\int \omega(x) = \sum_i dt_i Z_i$. If one looks at $\exp(\int \omega(x)) =$



$\sum \frac{1}{n!}(\sum dt_i Z_i)^n = \sum \frac{1}{n!}[(dt_0 Z_0)^n + \dots]$

one sees products such as

$$(dt_5 Z_5)(dt_1 Z_1)(dt_2 Z_2)\dots$$

Such terms do not make sense, because we always need products

$$(dt_i Z_i)(dt_j Z_j)(dt_k Z_k)(dt_l Z_l)\dots$$

such that $t_i > t_j > t_k > t_l \dots$ The path ordering is exactly this! For instance for $n = 2$

$$\left(P\exp\int\dots\right) \simeq \underset{\nearrow \atop n!}{\frac{1}{2}} P\left((dt_0 Z_0)^2 + (dt_1 Z_1)(dt_0 Z_0) + (dt_0 Z_0)(dt_1 Z_1) + (dt_1 Z_1)^2\right)$$

$$= \frac{1}{2}\left[(dt_1 Z_1)(dt_0 Z_0) + (dt_1 Z_1)(dt_0 Z_0)\right] =$$

$$(dt_1 Z_1)(dt_0 Z_0)$$

as desired. The combinatorical factor $\frac{1}{n!}$ just match the number of factors in the expansion of $(a+b+c\dots)^n$.

<u>Comment</u>: For an abelian group the various Z_i commute, and P is not needed.

We now calculate the derivative (D) in the direction γ

$$(D) = \frac{\psi(y) - U(\gamma)\psi(x)}{\varepsilon} = \frac{\psi(y) - \psi(x) - Z\psi(x)\cdot\varepsilon}{\varepsilon}$$

$$= X^i(x)\frac{\partial\psi(x)}{\partial x^i} - \underbrace{Z(X)}_{Z_i X^i}\psi(x) = X\left(\frac{\partial\psi}{\partial x} - Z\psi\right) \equiv X\cdot\nabla\psi$$

To summarize:

A covariant derivative for a PFB has the form

$$\nabla = \partial - Z$$

where $Z = \sum' Z_i(x)\, dx_i$ is a 1-form on M with values in the Lie algebra of G:

$$Z_i = \sum_a \Omega_i^{\ a}(x)\, T_a$$

where the T_a are a basis of $d\pi(\mathfrak{g})$. It is now natural to identify Z with the gaug fields A!!!

We have considered gauge transformations, $\psi_W(x) = g_{uw}^{-1}\, \psi_u(x)$

We write instead for a gauge transformation

$$\psi(x) \longrightarrow U(h_x)\, \psi(x)$$

which is a transformation $U: V_x \to V_x$, and thus an Element of $\pi(g)$ and $h: x \to h_x$, where $x \in M$ and $h_x \in G$.

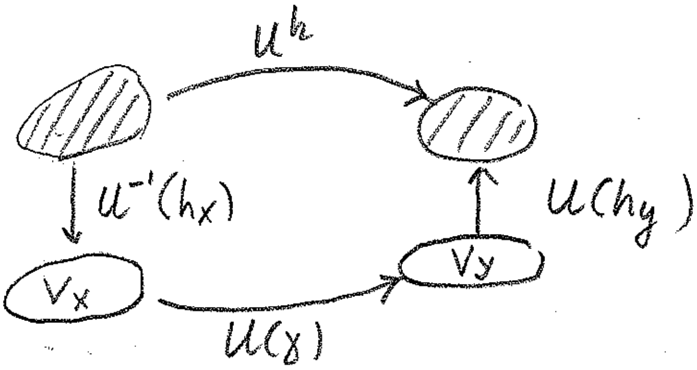
On the other hand we have

$$U(\gamma): V_x \to V_y$$

In the "gauge transformed" system we then have

$$\overset{h}{U}(\gamma):\ U(h_x) V_x \longrightarrow U(h_y) V_y$$

$$U^h(\gamma) = U(h_y)\, U(\gamma)\, \underbrace{U^{-1}(h_x)}_{U(h_x^{-1})}$$



Therefore we associate to $\mathcal{U}^h(\gamma)$ a gauge field

$$\mathcal{U}^h(\gamma) \sim \pi(h_y)(1+\varepsilon Z)\pi(h_x^{-1}) \equiv 1+\varepsilon Z^h$$

$$\pi(h_y) = \pi(h_x) + \varepsilon \frac{\partial \pi}{\partial x}(h_x)$$

$$\to 1+\varepsilon Z^h = 1 + \varepsilon\Big(\underbrace{\frac{\partial\pi}{\partial x}(h_x)\,\pi(h_x^{-1}) + \pi(x)\, Z\, \pi(h_x^{-1})}_{Z^h}\Big)$$

$$Z^h = \pi(x)Z(x)\pi(x^{-1}) - \pi(x)\frac{\partial}{\partial x}\pi(h_x^{-1})$$

$\Big($ since $\frac{\partial}{\partial x}(\pi\cdot\pi^{-}) = 0 = \frac{\partial \pi}{\partial x}\pi^{-} + \pi\frac{\partial\pi^{-}}{\partial x}$ $\Big)$

We can rewrite this as ($Z \sim$ gauge field)

$$A \longrightarrow g A g^{-1} - g\,\partial g^{-1}$$

$g \in G$, $A \in \mathfrak{g}$. This formula is the usual one for gauge transformations of gauge fields.

We call Z (or A) an affine connection; it is a way to "connect" the vector spaces at x and y.

We can also calculate the change of ψ if we go from x to y' in two different ways.

$$y = x + \varepsilon X \qquad x = y''' = y'' - \varepsilon Y$$
$$y' = y + \varepsilon' Y \qquad X = X_i \frac{\partial}{\partial x^i}$$
$$y'' = y' - \varepsilon X = x + \varepsilon' Y$$

$$\psi(y) = \psi(x) + \varepsilon \nabla_X \psi = \psi(x) + \varepsilon X_i(\partial_i + A_i)\psi$$

$$\psi(y+\varepsilon' Y) = \psi(x) + \varepsilon\nabla_X\psi + \varepsilon'\nabla_Y\psi(x) + \varepsilon\varepsilon'\nabla_Y\nabla_X\psi(x)$$

$$\psi(y+\varepsilon' Y) = \psi(x) + \varepsilon'\nabla_Y\psi + \varepsilon\nabla_X\psi(x) + \varepsilon\varepsilon'\nabla_X\nabla_Y\psi(x)$$

$$-\left(\psi_{\lrcorner\uparrow} - \psi_{\ulcorner\rightarrow}\right) = \varepsilon\varepsilon'\left(\nabla_y\nabla_x - \nabla_x\nabla_y\right)\psi$$

$$= X_i Y_j\Big[\left(\partial_i\partial_j - \partial_j\partial_i\right) + \left(Z_i Z_j - Z_j Z_i\right) X_i X_j$$
$$+ \left(\partial_i Z_j - \partial_j Z_i\right)\Big]\ \psi$$

$$\equiv X_i Y_j\ F_{ij}\ \psi$$

$$\boxed{F_{ij} = \left(\partial_i Z_j - \partial_j Z_i\right) + [Z_i, Z_j]}$$

This is of course the field strength. It is a function on the the space $TM \times TM$, thus is a 2-Form, which is called the curvature.

Geodesics

geodesics are curves of zero acceleration (straight line, "shortest lines". It is interesting to note that the motion in M of a charged particle is "more" geodesic in P.

Lit: David Bleecker
Gauge theory and variational principle