Lecture 18

Transport in metals

Drude model

The simplest classical model for transport is the Drude model. In its simplest version we assume the Newton's law for the average carrier velocity

$$m \frac{d\vec{v}}{dt} = -\frac{m\vec{v}}{\tau} + \vec{f}$$

Here f is the force due to electric and magnetic field and $-\frac{mv}{\tau}$ term describes collisions. In static case with only electric field present $\vec{f} = -e\vec{E}$ and

$$\vec{v} = -\frac{eE\tau}{m}$$

Electric current is given by simple flow of electrons

$$\vec{j} = -ne\vec{v}$$

and we obtain Drude formula for conductivity

$$\vec{j} = \sigma \vec{E}, \quad \sigma = \frac{ne^2\tau}{m}$$

If magnetic field is present

$$\vec{f} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right)$$

And the force balance is

$$\frac{m v}{\tau} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right)$$

Multiplying by $-ne$ and introducing the current $\vec{j} = -en\vec{v}$ we obtain

$$\frac{m\vec{j}}{\tau} = ne^2\vec{E} - \frac{e(\vec{j} \times \vec{H})}{c}$$

Solving it for electric field

$$\vec{E} = \frac{m}{ne^2\tau}\vec{j} + \frac{\vec{j} \times \vec{H}}{nec}$$

$$E_x = \rho_{xx} j_x + \rho_{xy} j_y$$

Longitudinal resistivity doesn't change when magnetic field is present

But now we have Hall resistivity

$$\rho_{xy} = \frac{B}{nec} \qquad \frac{\rho_{xy}}{\rho_{xx}} = \tan\theta_H = \omega_c\tau, \ \omega_c = \frac{eH}{mc}$$

We can invert $E(j)$ equation and get the longitudinal and Hall conductivity

$$\sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2} = \frac{ne^2\tau}{m} \frac{1}{1+(\omega_c\tau)^2}$$

$$\sigma_{xy} = -\sigma_{yx} = \frac{\rho_{xy}}{\rho_{xx}^2 + \rho_{xy}^2} = \frac{ne^2\tau}{m} \frac{\omega_c\tau}{1+(\omega_c\tau)^2}$$

To describe AC response we introduce $E = E_\omega e^{-i\omega t}$

Then equation of motion reads

$$-i\omega m\vec{v}_\omega = -m\frac{\vec{v}_\omega}{\tau} + \vec{f}_\omega \quad \text{or}$$

$$m\vec{v}_\omega\left(\frac{1}{\tau} - i\omega\right) = \vec{f}_\omega$$

Thus in all the formulas for conductivity (resistivity) we simply should replace $\frac{1}{\tau} \to \frac{1}{\tau} - i\omega$

In the absence of magnetic field

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \sigma_0 = \frac{ne^2\tau}{m}$$

For electric field varying also in space one can write general linear response

$$j(\vec{q}, \omega) = \sigma(\vec{q}, \omega) E(q, \omega)$$

Using Maxwell equations we can relate σ and ε. Indeed without external charges

$$\operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0$$

$$\operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

$$\operatorname{rot} \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \quad \vec{j} = \sigma \vec{E}$$

Then $\operatorname{rot}\operatorname{rot} \vec{E} = -\nabla^2 E = \frac{i\omega}{c}[\vec{\nabla} \times \vec{H}] =$

$$= i\omega\left(\frac{4\pi}{c}\sigma\vec{E} - \frac{i\omega}{c}\vec{E}\right)$$

And

$$-\nabla^2 E = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi i \sigma}{\omega}\right)\vec{E}$$

which is the usual wave equation

$$-\nabla^2 \vec{E} = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,\vec{E}$$

with a complex dielectric constant

$$\varepsilon(\omega) = 1 + \frac{4\pi i \sigma(\omega)}{\omega}$$

For high frequencies $\omega\tau >> 1$

the Drude expression for conductivity reproduces plasma oscillations

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{with } \omega_p^2 = \frac{4\pi n e^2}{m}$$

Since we derive this $\varepsilon(\omega)$ from more microscopic considerations we can argue that $\sigma(\omega) = \frac{i\omega_p^2}{4\pi\omega}$ for $\omega \to \infty$ is more general result than the Drude approximation

Being response functions $\sigma(\omega)$ and $\varepsilon(\omega)$ satisfy the Kramers-Kronig relation

For $\sigma(\omega) \equiv \sigma_1(\omega) + i\,\sigma_2(\omega)$

real and imaginary parts are related

$$\sigma_1(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\sigma_2(\omega')}{\omega - \omega'}$$

$$\sigma_2(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\sigma_1(\omega')}{\omega - \omega'}$$

Since for $\omega \to \infty$ $\sigma_2(\omega) \to \frac{\omega_p^2}{4\pi\omega}$

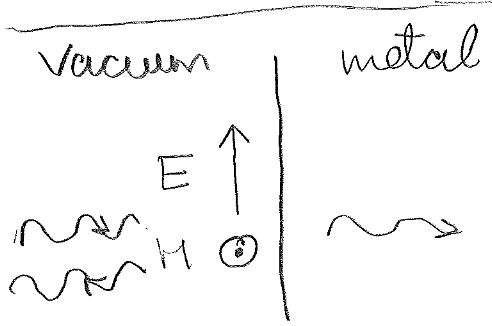
taking ω out of the second Kramers-Kronig relations we obtain the sum rule

$$\int_0^{\infty} d\omega' \sigma_1(\omega') = \frac{1}{2} \int_{-\infty}^{\infty} d\omega' \sigma_1(\omega') = \frac{\omega_p^2}{8} = \frac{\pi e^2 n}{2m}$$

Check that this sum rule is valid in the Drude approximation.

Reflectivity of a metal

Consider the light propagation in the sample



In the metal electric field is determined from the wave equation

$$-\nabla^2 \vec{E} = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,\vec{E}$$

with $\varepsilon(\omega)$ given by the Drude form

Note that ε has real and imaginary parts. In the vacuum we have the same wave equation but with $\varepsilon = 1$

Boundary conditions are obtained from

$\mathrm{rot}\,E = -\frac{1}{c}\frac{\partial H}{\partial t}$ and continuity of magnetic field. Then we obtain that the parallel to the surface component of electric field and its derivative should be continuous $E_1 = E_2$, $\frac{\partial E_1}{\partial x} = \frac{\partial E_2}{\partial x}$

Introducing electric field in the vacuum

$$E_v = \left(e^{ikx} + r e^{-ikx}\right) e^{-i\omega t}$$ and in the metal

$$E_m = t\, e^{ik_1x - \lambda x} e^{-i\omega t}$$ we obtain

$$(\lambda - ik_1)^2 = -\frac{\omega^2}{c^2}\varepsilon(\omega), \quad k^2 = \frac{\omega^2}{c^2} \Rightarrow$$

$$(\lambda - ik_1) = -ik\sqrt{\varepsilon(\omega)}$$

Making use of the boundary conditions

$$\begin{cases} t = 1 + r \\ t\,(ik_1 - \lambda) = ik\,(1 - r) \end{cases}$$

and reflectivity amplitude

$$r = \frac{i(k - k_1) + \lambda}{i(k + k_1) - \lambda}$$

Then the reflectivity

$$R = |r|^2 = \frac{\lambda^2 + (k - k_1)^2}{\lambda^2 + (k + k_1)^2}$$

Let us analize light propagation for different frequencies. Drude expression for dielectric constant

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2 \tau i}{\omega(1 - i\omega\tau)} = 1 - \frac{\omega_p^2 \tau^2}{1+\omega^2\tau^2} + \frac{i}{\omega}\frac{\omega_p^2 \tau}{1+\omega^2\tau^2}$$

For small frequencies $\omega\tau \ll 1 \ll \omega_p \tau$

$$\varepsilon = \varepsilon_1 + i\,\varepsilon_2 \,, \quad \varepsilon_1(\omega) = 1 - \omega_p^2\tau^2 \approx -\omega_p^2\tau^2$$

$$\varepsilon_2(\omega) = \frac{\omega_p^2 \tau}{\omega} \gg |\varepsilon_1(\omega)|$$

In this case ε is practically imaginary, electric field decays inside the metal within the skin depth $\delta(\omega) = \left(\frac{c^2}{2\pi\sigma\omega}\right)^{1/2} \propto \frac{1}{\sqrt{\omega}}$

$$E \propto e^{-\frac{x}{\delta} + \frac{ix}{\delta}}$$

In this case $\lambda = K_1 = K\sqrt{\frac{2}{\omega\tau}} \gg K$

and reflectivity $R \to 1$

For $1 \ll \omega\tau \ll \omega_p\tau$

$$\varepsilon(\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2} + i\,\frac{\omega_p^2}{\omega^3\tau}$$

In this case $\varepsilon_1 \approx -\frac{\omega_p^2}{\omega^2}$

$\varepsilon_2 \ll \varepsilon_1$

In this case field decays inside the sample within $\delta = \frac{c}{\omega_p}$,

$\lambda \gg K_1, K$ and reflectivity again is very close to unity

The situation changes above the plasma frequency. For $\omega > \omega_p$ (ultraviolet range)

$$\varepsilon_1 = 1 - \frac{\omega_p^2}{\omega^2} > 0 \;,\; \varepsilon_2 \ll \varepsilon_1$$

Then $K_1 \doteq K\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \gg \lambda$

In this case reflectivity

R quickly drops to very small values

$$R \propto \frac{\omega_p^2}{\omega^2} \quad \text{for} \quad \omega \gg \omega_p$$

This behavior explains color of metals

Plasma frequency ω_p lies well above visible light $\hbar\omega \sim 1.5 - 3.5\,eV$. Then all the visible frequencies are completely reflected and metal looks shiny white. Some metals have color (like gold - yellow, copper - reddish). For these metals reflectivity is reduced in the green - blue range. This is not related to plasma frequency but is due to the interband transitions.

$3d \to 4s$ in case of Cu