Symmetries in Physics

Problem Sets

ETH Zurich, HS15

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Contents

1.1. Defining a group

Consider the following minimalistic definition of a group:
A group is a set G together with a map

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G}, \qquad (g, h) \mapsto gh \tag{1.1}$$

satisfying the following axioms:

- *associativity:*
$$a(bc) = (ab)c \qquad \text{for all } a, b, c \in \mathrm{G}; \tag{1.2}$$
- *unit element:* there exists an $e \in \mathrm{G}$ such that
$$ea = a \qquad \text{for all } a \in \mathrm{G}; \tag{1.3}$$
- *inverse:* for all $a \in \mathrm{G}$ there exists an $a^{-1} \in \mathrm{G}$ such that
$$a^{-1}a = e. \tag{1.4}$$

a) Show, using only the axioms above, that $aa^{-1} = e$. Using this show that $ae = a$ and finally that the unit is unique.

b) Replace the unit axiom (1.3) above with

$$ae = a \qquad \text{for all } a \in \mathrm{G}. \tag{1.5}$$

Find a set that fulfils the modified axioms (1.2), (1.5), (1.4) and is not a group.
Hint: Think of a subset of 2×2 matrices.

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1.2. Familiarising with groups

Consider the following sets and establish whether they form a group or not.

a) The set of all non-zero real numbers with ordinary multiplication as the group operation. What changes if we include also the element zero?

b) The set of all real numbers (including zero) with ordinary addition as the group operation.

c) The set of permutations S_n acting on the set of n symbols $A_n := \{1, 2, \ldots, n\}$. What is the number of elements, i.e. the order, of S_n?

Consider in particular S_3. All its elements can be generated by the iterated products of two elements σ_1 and σ_2 satisfying the conditions

$$\sigma_1^2 = \sigma_2^2 = e, \tag{1.6}$$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2. \tag{1.7}$$

d) Find all the elements of S_3, and argue that no further elements are generated.

e) Find a suitable action for σ_1, σ_2 on the set A_3.

f) Find a 3-dimensional representation for these elements.
Hint: Think of A_3 as a basis for the 3-dimensional representation space.

g) *optional:* Find a non-trivial 2-dimensional representation for these elements.

h) *optional:* Find a non-trivial 1-dimensional representation for these elements.

2.1. Maps between groups

For the whole exercise let G and H be groups and $g, g_1, g_2 \in \mathrm{G}$ and $h, h_1, h_2 \in \mathrm{H}$.

a) A *group homomorphism* is a map $\varphi : \mathrm{G} \to \mathrm{H}$ that respects the group structure:

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) \qquad \text{for all } g_1, g_2 \in \mathrm{G}. \tag{2.1}$$

Show that $\varphi(e_\mathrm{G}) = e_\mathrm{H}$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$, where e_G and e_H denote the identity elements of G and H, respectively.

b) A bijective group homomorphism is called *group isomorphism*. Consider the set of *automorphisms* of a group G defined as

$$\mathrm{Aut}(\mathrm{G}) := \{\varphi : \mathrm{G} \to \mathrm{G}; \varphi \text{ is a group isomorphism}\}. \tag{2.2}$$

Show that Aut(G) together with the composition of maps $\circ$ forms a group.

c) Show that the *direct product* $\mathrm{G} \times \mathrm{G}$ defined with multiplication

$$(g_1, h_1)(g_2, h_2) := (g_1 g_2, h_1 h_2) \tag{2.3}$$

is a group.

d) Let G act on G by a group homomorphism $\varphi : \mathrm{G} \to \mathrm{Aut}(\mathrm{H})$, in particular $\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2)$ and $\varphi(g)(h_1 h_2) = \varphi(g)(h_1)\varphi(g)(h_2)$.
Show that the *semi-direct product* $\mathrm{G} \rtimes_\varphi \mathrm{G}$ defined with the multiplication

$$(h_1, g_1)(h_2, g_2) := \big(h_1 \varphi(g_1)(h_2), g_1 g_2\big) \tag{2.4}$$

is again a group.

2.2. The group SO(3)

Consider the set of proper rotations in three dimensions

$$\mathrm{SO}(3) := \{A \in \mathrm{End}(\mathbb{R}^3); A^\mathsf{T} A = \mathrm{id}, \det(A) = 1\}. \tag{2.5}$$

a) Show that SO(3) together with the matrix multiplication forms a group.

b) Show that each element $A \in \mathrm{SO}(3)$ is a rotation about some axis, i.e. show that for all $A \in \mathrm{SO}(3)$ there exists a $\vec{v} \in \mathbb{R}^3$ such that $A\vec{v} = \vec{v}$.

Hint: When does $A - \mathrm{id}$ have a non-trivial kernel?

c) Fix a vector $\vec{v} \in \mathbb{R}^3 \setminus \{0\}$. The *stabiliser subgroup* of SO(3) with respect to $\vec{v}$ is defined as

$$\mathrm{SO}(3)_{\vec{v}} := \{A \in \mathrm{SO}(3); A\vec{v} = \vec{v}\}. \tag{2.6}$$

For a given $A \in \mathrm{SO}(3)$ show that

$$\mathrm{SO}(3)_{A\vec{v}} = A\,\mathrm{SO}(3)_{\vec{v}}\,A^{-1} := \{ARA^{-1}; R \in \mathrm{SO}(3)_{\vec{v}}\} \tag{2.7}$$

and furthermore that $\mathrm{SO}(3)_{\vec{v}} \equiv \mathrm{SO}(2)$.

d) Consider a cube centred at the origin. How many rotations are there that map the cube to itself.

3.1. From curves to Lie algebras

Let G be a group and $A(t)$ a differentiable curve on G with $A(0) = 1$, where 1 is the identity element of G. Then the derivative

$$a = \frac{d}{dt} A(t)\bigg|_{t=0} \tag{3.1}$$

defines an element of the tangent space $\mathfrak{g}$ of the identity.

a) Show that the set of all derivatives of such curves defines a real vector space.

b) Since $\mathfrak{g}$ is therefore a vector space, show that

$$\frac{d}{dt}\big(A_1(t)\, a_2\, A_1(t)^{-1}\big)\bigg|_{t=0} = a_1 a_2 - a_2 a_1 \in \mathfrak{g}, \qquad a_1 = \frac{d}{dt} A_1(t)\bigg|_{t=0} \tag{3.2}$$

and therefore, the Lie algebra $\mathfrak{g}$ is endowed with a bilinear product

$$[\![a_1, a_2]\!] := a_1 a_2 - a_2 a_1 = a_3 \in \mathfrak{g}. \tag{3.3}$$

Hint: Show in particular that

$$\frac{d}{dt}\big(A(t)^{-1}\big)\bigg|_{t=0} = -a. \tag{3.4}$$

c) Consider now the group G = SO(3). Starting from the properties of the curves $R(t)$ on G, construct the corresponding Lie algebra and find an explicit basis.

d) Starting again from the notion of curves, construct the Lie algebra corresponding to the SU(2) group and find a basis for it.

Can you find a relationship between the $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ algebras?

3.2. Euler angles

Any rotation in three dimensions $R \in \mathrm{SO}(3)$ can be expressed as

$$R = R^z_\phi R^y_\theta R^z_\psi \tag{3.5}$$

in terms of the three Euler angles $0 \le \psi, \phi < 2\pi$, $0 \le \theta \le \pi$ and the rotation matrices around the z and y axes

$$R^y_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R^z_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

a) Find the Euler angles for the rotation around the x-axis $0 \le \alpha < 2\pi$

$$R^x_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \tag{3.7}$$

b) Can you derive the Lie algebra $\mathfrak{so}(3)$ from this form of R?

Hint: Consider either commutators or cylindrical coordinates for the xy-axes.

4.1. The Lie algebra $\mathfrak{so}(4)$

a) Determine the generators and their commutators for the Lie algebra $\mathfrak{so}(4)$. Here $\mathfrak{so}(4)$ is the Lie algebra associated to the group SO(4) consisting of real orthogonal 4×4 matrices with determinant 1.

b) Show that, as a real Lie algebra, $\mathfrak{so}(4) \equiv \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, where the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is defined to be the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces with the requirement that $[\![\mathfrak{g}_1, \mathfrak{g}_2]\!] = 0$.

4.2. BCH formula

The Baker–Campbell–Hausdorff (BCH) formula states that

$$\exp(A) \cdot \exp(B) = \exp(A \star B), \tag{4.1}$$

where $A \star B$ is an element of the Lie algebra generated by A and B and equals

$$A \star B = A + B + \tfrac{1}{2}[\![A, B]\!] + \tfrac{1}{12}[\![A, [\![A, B]\!]]\!] + \tfrac{1}{12}[\![B, [\![B, A]\!]]\!] + \ldots\ . \tag{4.2}$$

Prove the BCH formula to this order assuming that A, B are matrices.

Hint: Replace Ω_i by $t\Omega_i$ ($i = 1, 2$) and expand both sides to the appropriate order in t.

4.3. Lorentz transformations

The Lorentz transformations are the coordinate transformation that leave the Minkowski metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$ and corresponding distance squared $x^{\mathsf{T}}\eta x = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ invariant, i.e.

$$\mathrm{SO}(3,1) := \big\{A \in \mathrm{Aut}(\mathbb{R}^4);\, A^{\mathsf{T}}\eta A = \eta\big\}. \tag{4.3}$$

Find its Lie algebra, determine a set of generators, interpret them and find their commutation relations.

5.1. Irreducible representations of abelian groups

Using Schur's lemma, show that the irreducible representations of a finite abelian group G are all one-dimensional.

5.2. Eigenvalues of representations of finite groups

Consider a finite-dimensional representation ρ of a finite group G. Show that the eigenvalues of $\rho(g)$ are roots of unity for any $g \in \mathrm{G}$.

5.3. Orbit-Stabiliser Theorem

Given a group action on a set X, we define the stabiliser of an element $x \in X$ as the subset of transformations that map x onto itself,

$$\mathrm{G}_x := \big\{g \in \mathrm{G}; g\cdot x = x\big\}. \tag{5.1}$$

The orbit-stabiliser theorem states that the order $|\mathrm{G}|$ of the group G can be calculated as the product of the order of the stabiliser of x times the cardinality $|X_x|$ of the orbit $X_x = \mathrm{G}x := \{g\cdot x; g \in \mathrm{G}\}$

$$|X_x|\cdot|\mathrm{G}_x| = |\mathrm{G}| \ . \tag{5.2}$$

This is true for any $x \in X$.

a) Consider the symmetry group O of the cube. List the elements of O in terms of transformations of the cube. Consider only proper rotations, do not include reflections.

b) Verify the orbit-stabiliser theorem by considering the action of O on

- the set F of faces of a cube;
- the set V of vertices of a cube.

c) Prove the orbit-stabiliser theorem in the general case.
Hint: fix an element $x \in X$, then

- show that the relation $g \sim h$ if $g\cdot x = h\cdot x$ is an equivalence relation;
- show that the number of elements of G in each equivalence class is equal and compute it;
- show that the number of equivalence classes into which G is partitioned via $\sim$ is equal to the cardinality of X_x.

6.1. Characters and irreducible representations

Given a representation ρ_A and an irreducible sub-representation ρ_k, the multiplicity with which ρ_k appears within ρ_A is given by the hermitian inner product $\langle \chi_A, \chi_k \rangle$ defined by

$$\langle \chi_A, \chi_B \rangle := \frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \chi_A(g)^* \, \chi_B(g). \tag{6.1}$$

a) Show that characters of inequivalent irreducible representations are mutually orthogonal.

b) Show that a representation ρ_A is irreducible if and only if $\langle \chi_A, \chi_A \rangle = 1$.

c) Using these results, show that the tensor product of any irreducible representation with a one-dimensional representation is another irreducible representation.

6.2. Combinatorics of a class of labelled graphs

In this exercise we want to find connected graphs with N nodes labelled by strictly positive integers d_i, $i = 1, \ldots, N$, without a common divisor, such that the label of each node equals half the sum of labels of the adjacent nodes:

$$2d_i = \sum_{j \sim i} d_j \qquad \text{for all } i = 1, \ldots, N. \tag{6.2}$$

The sum runs over all neighbouring nodes j (nodes directly connected to node i).

a) Show that within a linear chain of nodes, the labels must depend linearly on the position within the chain.

b) Show that if there is a cycle then all nodes are part of the cycle and $d_i = 1$ for all i.

c) Show that no node has more than 4 neighbours. Furthermore, there is a unique graph with a 4-valent node.

The remaining graphs have the structure of a tree with leaves (1-valent nodes) and vertices (3-valent nodes) joined by linear chains (of 2-valent nodes).

d) Show that every leaf connects to a vertex via a chain and find a relationship between the two corresponding labels.

e) Show that there are three distinct graphs with a single vertex.

f) In a graph with more than one 3-vertex, show that the vertex with the smallest label can be linked (via linear chains) to at most one other vertex and to at most one leaf. The only remaining graph must therefore have two 3-vertices with equal labels.

g) Show that there is a unique graph with multiple links (where the summands in the equation are weighted appropriately).

7.1. Dihedral group

The dihedral group D_n is generated by two elements d and s satisfying the relations

$$d^n = s^2 = e, \qquad d^{-k} s = s d^k \quad \text{for all } k \in \mathbb{Z}. \tag{7.1}$$

a) The simplest dihedral group is D_3, the symmetry group of an equilateral triangle. Identify the elements d and s with symmetries of the triangle and verify that the relations (7.1) are satisfied. Compute the multiplication table of D_3.

b) Suppose that S and D are $l \times l$ matrices satisfying

$$D^n = S^2 = \mathrm{id}_l, \qquad S\, D^k\, S = D^{-k} \quad \text{for all } k \in \mathbb{Z}. \tag{7.2}$$

Show that $\rho(s) = S$ and $\rho(d) = D$ then defines an l-dimensional representation of D_n (acting by $l \times l$ matrices on an l-dimensional vector space).

c) Find all the irreducible representations of the dihedral group D_n, distinguishing between the cases n even and n odd.

Hint: All irreducible representations of D_n have dimension 1 or 2. Remember the relation

$$|\mathrm{G}| = \sum_{i=1}^{m} (\dim \mathbb{V}_i)^2, \tag{7.3}$$

where the sum runs over the inequivalent irreducible representations of the finite group G. Find sufficiently many inequivalent irreducible representations such that this formula is satisfied.

d) Decompose the tensor products of all irreducible representations.

8.1. Three-dimensional Lie algebras

Consider the following three-dimensional complex Lie algebras defined in terms of generators x, y, z and the commutation relations

1. $[\![x,y]\!] = 0, \quad [\![x,z]\!] = 0, \quad [\![y,z]\!] = 0,$ (abelian algebra)
2. $[\![x,y]\!] = z, \quad [\![x,z]\!] = 0, \quad [\![y,z]\!] = 0,$ (Heisenberg algebra)
3. $[\![x,y]\!] = x, \quad [\![x,z]\!] = 0, \quad [\![y,z]\!] = 0,$ (direct product)
4. $[\![x,y]\!] = y, \quad [\![x,z]\!] = y+z, \quad [\![y,z]\!] = 0,$
5. $[\![x,y]\!] = y, \quad [\![x,z]\!] = -z, \quad [\![y,z]\!] = x.$ ($\mathfrak{sl}(2)$)

Calculate the following for each Lie algebra:

a) Every Lie algebra acts on itself via the adjoint action $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined as $\mathrm{ad}(g)h := [\![g,h]\!]$. In particular, $\rho(g) := \mathrm{ad}(g)$ defines a representation, the *adjoint representation*.

Find the adjoint representation in matrix form.

b) Find the Killing form defined as $\kappa(g,h) := \mathrm{tr}(\mathrm{ad}(g)\,\mathrm{ad}(h))$ in the x, y, z basis.

c) Find the derived algebra $\mathfrak{g}' := [\![\mathfrak{g},\mathfrak{g}]\!] = \{[g,h]; g,h \in \mathfrak{g}\}$ and the derived algebra $\mathfrak{g}''$ of $\mathfrak{g}'$.

d) Which of these algebras is simple, i.e. has no non-trivial ideal?

An *ideal* is a subalgebra $\mathfrak{i} \subset \mathfrak{g}$ such that $[\![\mathfrak{g},\mathfrak{i}]\!] \subset \mathfrak{i}$.

8.2. Cartan–Weyl basis of $\mathfrak{sl}(3)$

Write out explicitly the Lie brackets of $\mathfrak{sl}(3)$ in the Cartan–Weyl basis.

9.1. Simple Lie groups and simple Lie algebras

A simple Lie group is a connected non-abelian Lie group with no proper connected normal subgroups (a subgroup H of a group G is called *normal* if for all $h \in \mathrm{H}$ and $g \in \mathrm{G}$, $ghg^{-1} \in \mathrm{H}$). We want to understand what this condition means in terms of the Lie algebra of the group.

In this exercise, we will always consider compact connected Lie groups, for which the exponential map is onto, i.e. each $g \in \mathrm{G}$ can be written as $g = \mathrm{e}^A$ for some $A \in \mathfrak{g}$.

a) A *subalgebra* is a subspace of an algebra which is closed under multiplication. In the case of a Lie algebra this means that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ obeys

$$\mathfrak{h} \subset \mathfrak{g}, \quad [\![\mathfrak{h}, \mathfrak{h}]\!] \subset \mathfrak{h}. \tag{9.1}$$

Show that the exponential map maps a subalgebra $\mathfrak{h}$ into a subgroup H of G.

b) An *ideal* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the property that

$$[\![\mathfrak{g}, \mathfrak{h}]\!] \subset \mathfrak{h}. \tag{9.2}$$

A proper ideal of $\mathfrak{g}$ is an ideal which is neither trivial nor all of $\mathfrak{g}$. Show, using the exponential map, that a Lie group is simple if and only if its Lie algebra contains no proper ideals.

10.1. Representation of $\mathfrak{su}(3)$

Construct explicitly the finite-dimensional irreducible representation $\rho : \mathfrak{su}(3) \to \mathrm{End}(\mathbb{V})$ that is generated from the highest-weight state $|\mu\rangle$ satisfying

$$\rho(L_{12})|\mu\rangle = \rho(L_{13})|\mu\rangle = \rho(L_{23})|\mu\rangle = 0, \tag{10.1}$$

with the weight given by ($H_{jk} := L_{jj} - L_{kk}$)

$$\rho(H_{12})|\mu\rangle = 2|\mu\rangle, \qquad \rho(H_{23})|\mu\rangle = 0. \tag{10.2}$$

Determine, in particular, the dimension of ρ and the eigenvalues (with multiplicities) of $\rho(H_{12})$ and $\rho(H_{23})$. Proceed as follows:

a) Show that $\mathbb{V}$ is spanned by the vectors $W|\mu\rangle$, where W is any word in $\rho(L_{21})$ and $\rho(L_{32})$.

b) Acting on $|\mu\rangle$ with $\rho(L_{21})$ and $\rho(L_{32})$, construct a basis of ρ. For any new vector, compute its eigenvalues under $\rho(H_{12})$ and $\rho(H_{23})$, and verify that you can go back to the vectors previously constructed (and thus, by recursion, to $|\mu\rangle$), by acting with some element of $\mathfrak{su}(3)$. If this is not possible the vector must be 0, since by assumption the representation is irreducible.

10.2. Representation theory of $\mathfrak{sl}(5, \mathbb{C})$

Develop the representation theory of the complexification $\mathfrak{sl}(5, \mathbb{C})$ of the Lie algebra $\mathfrak{su}(5)$.

a) Identify the Cartan subalgebra $\mathfrak{h}$ and define a suitable basis for the dual space $\mathfrak{h}^*$. Find the roots of the algebra and describe them in terms of this basis $\mathfrak{h}^*$.

b) Identify subalgebras $\mathfrak{sl}(2, \mathbb{C})$ inside $\mathfrak{sl}(5, \mathbb{C})$, and deduce the structure of the possible weights of any finite-dimensional representation of $\mathfrak{sl}(5, \mathbb{C})$.

c) Choose a linear functional on the dual Cartan subalgebra, and divide the roots into positive and negative roots. Then identify the possible highest weights of any finite dimensional representation. Show that these highest weights can be labelled by four non-negative integers

11.1. Young Tableaux *optional:*

This week's class will give a basic introduction to Young tableaux for the representation theory of $\mathrm{SU}(N)$. You may familiarise yourself with this topic, see chapter 7 of the lecture notes.

12.1. Algebra isomorphism

Consider the algebra $\mathfrak{g}^{(M)}$ defined by

$$\mathfrak{g}^{(M)} := \{A \in \mathfrak{gl}(n, \mathbb{C}); A^\intercal M = -MA\}, \tag{12.1}$$

where M is an invertible $n \times n$ matrix. We want to show that $\mathfrak{g}^{(M)} \equiv \mathfrak{so}(n, \mathbb{C})$ if M is symmetric.

a) Prove that $\mathfrak{g}^{(M)}$ is a Lie algebra.

b) Consider a matrix $T = P^\intercal M P$ for some invertible matrices P and M. Show that the Lie algebras $\mathfrak{g}^{(M)}$ and $\mathfrak{g}^{(T)}$ are isomorphic.

c) Prove that $\mathfrak{g}^{(M)} \equiv \mathfrak{so}(n, \mathbb{C})$ for a symmetric M.

d) *optional:* Show that $[M^{-1}M^\intercal, A] = 0$ for all $A \in \mathfrak{g}^{(M)}$ and argue why one should choose M to be either symmetric or anti-symmetric.

12.2. Dynkin diagram of $\mathfrak{so}(2r + 1, \mathbb{C})$

Consider the algebra $\mathfrak{g}^{(S)}$ with $n = 2r + 1$ as defined in problem 12.1, where S is the invertible symmetric matrix in $(r, r, 1)$ block form

$$S := \begin{pmatrix} 0 & \mathrm{id}_r & 0 \\ \mathrm{id}_r & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12.2}$$

a) Write the elements of $\mathfrak{g}^{(S)}$ as block matrices adapted to the blocks of S.

b) Let $\mathfrak{h}$ be the Cartan subalgebra, which is given by the space of diagonal matrices of the form

$$H := \sum_{i=1}^{r} h_i \,(L_{i,i} - L_{i+r,i+r}), \tag{12.3}$$

where the $L_{i,j}$ are matrices with 1 in row i and column j and 0 everywhere else. Find the generators L_α corresponding to the roots α and check that

$$[\![H_\alpha, L_\alpha]\!] \neq 0 \quad \text{with} \quad H_\alpha := [\![L_\alpha, L_{-\alpha}]\!]. \tag{12.4}$$

c) A basis for the dual Cartan subalgebra is then given by the simple roots

$$\beta_k := (L_{k,k} - L_{k+1,k+1} - L_{r+k,r+k} + L_{r+k+1,r+k+1})^*, \tag{12.5}$$
$$\beta_r := (L_{r,r} - L_{2r,2r})^*. \tag{12.6}$$

Determine the Cartan matrix and the corresponding Dynkin diagram.

13.1. Infinitesimal conformal transformations on $\mathbb{R}^{n,m}$

A conformal transformation $x^\mu \mapsto \tilde{x}^\mu$ rescales the metric $\eta_{\mu\nu}$ by some factor $\lambda \in \mathbb{R}$:

$$\eta_{\mu\nu} \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} = \lambda^2(x) \eta_{\alpha\beta}. \tag{13.1}$$

a) Write an infinitesimal conformal transformation as

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \epsilon \omega^\mu(x) + \mathcal{O}(\epsilon^2) \tag{13.2}$$

and find a differential equation for ω^μ.
Hint: Expand as well $\lambda^2(x) = 1 + \epsilon\chi(x) + \mathcal{O}(\epsilon^2)$ and collect first order terms.

b) Show that

$$\omega^\alpha = c^\alpha + m^\alpha{}_\beta x^\beta + \kappa x^\alpha + b^\alpha x^2 - 2b^\beta x_\beta x^\alpha \tag{13.3}$$

with $m_{\alpha\beta} = -m_{\beta\alpha}$ satisfies the differential equation found in part a).

Indeed, for $n + m > 2$ the most general solution is given by (13.3). It motivates the following definition of the Lie algebra generators and their action on $\mathbb{R}^{n,m}$:

$$\begin{aligned} \mathrm{P}_\mu x^\nu &= i\delta^\nu_\mu, & \mathrm{K}^\mu x^\nu &= i x^\mu x^\nu - \tfrac{i}{2}\eta^{\mu\nu} x^2, \\ \mathrm{D} x^\nu &= i x^\nu, & \mathrm{M}_{\mu\nu} x^\rho &= i\delta^\rho_\nu x_\mu - i\delta^\rho_\mu x_\nu. \end{aligned} \tag{13.4}$$

c) Determine their commutators.

d) Show that the conformal algebra is isomorphic to $\mathfrak{so}(n+1, m+1)$.