Exercise 1. ***Hamilton-Jacobi equation***

A particle of charge q is constrained to move in a plane under the influence of a central force potential (non-electromagnetic) $V = \frac{1}{2}kr^2$ and a constant magnetic field $\mathbf{B}$ perpendicular to the plane, so that the vector potential can be expressed as

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}. \tag{1}$$

(a) Set up the Hamilton-Jacobi equation for Hamilton's characteristic function in polar coordinates (r, θ).

Hint. Start with the Lagrangian in cartesian coordinates and change to polar after incorporating the magnetic term. Introduce canonical momenta as usual to obtain the Hamiltonian.

(b) Separate the equation and reduce it to quadratures. Discuss the motion if the canonical momentum p_θ is zero at $t = 0$.

Solution.

(a) We start from the system Lagrangian in cartesian coordinates, which reads

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}(\dot{\mathbf{x}} \cdot \mathbf{A}) - \frac{k}{2}(x^2 + y^2). \tag{S.1}$$

Inserting the vector potential

$$\mathbf{A} = \frac{1}{2}B(x\mathbf{e}_y - y\mathbf{e}_x) \tag{S.2}$$

we get

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{qB}{2c}(\dot{y}x - \dot{x}y) - \frac{k}{2}(x^2 + y^2). \tag{S.3}$$

We now go to polar coordinates,

$$\begin{aligned} x &= r\cos\theta \quad , \quad \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ y &= r\sin\theta \quad , \quad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned} \tag{S.4}$$

to obtain

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{qB}{2c}r^2\dot{\theta} - \frac{k}{2}r^2. \tag{S.5}$$

We introduce the canonical momenta

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{qB}{2c}r^2. \end{aligned} \tag{S.6}$$

The Hamiltonian is then

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \frac{1}{2m}p_r^2 + \frac{1}{2mr^2}\left(p_\theta - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2. \tag{S.7}$$

The Hamilton-Jacobi (H-J) equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta} - \frac{qB}{2c}r^2\right)^2 + \frac{1}{2}kr^2 - \frac{\partial S}{\partial t} = 0 \tag{S.8}$$

(b) Since θ is a cyclic coordinate the corresponding conjugate momentum must be constant. We look then for a solution of the form

$$S(r,\theta,E,\alpha,t)=f(r,E,\alpha)+\alpha\theta-Et \tag{S.9}$$

where E and α are constants of motion. The H-J equation becomes

$$\frac{1}{2m}\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\alpha-\frac{qB}{2c}r^2\right)^2+\frac{1}{2}kr^2=E \tag{S.10}$$

with solution

$$f(r)=\int^r dz\,\sqrt{2mE-mkz^2-\frac{1}{z^2}\left(\alpha-\frac{qB}{2c}z^2\right)^2}. \tag{S.11}$$

If α (i.e, p_θ) is zero at $t=0$, the solution simplifies to

$$f(r)=\int^r dz\,\sqrt{2mE-m^2(\omega_0^2+\omega_c^2)z^2} \tag{S.12}$$

which is the harmonic oscillator problem with rescaled frequency $\omega=\sqrt{\omega_0^2+\omega_c^2}$. The frequencies are

$$\begin{aligned}\omega_0&=\sqrt{\frac{k}{m}} &&\text{natural harmonic oscillator frequency}\\ \omega_c&=\frac{qB}{2mc} &&\text{half cyclotron frequency.}\end{aligned} \tag{S.13}$$

Exercise 2. *Poisson brackets in the Kepler problem*

Show that the components of the Laplace-Runge-Lenz vector

$$\mathbf{A}=\frac{1}{\mu}\mathbf{p}\times\mathbf{L}-\frac{\mathbf{r}}{r} \tag{2}$$

satisfy the following relations

$$\{A_i,A_j\}=-\frac{2E}{\mu^2}\varepsilon_{ijk}L_k \tag{3}$$

where E is the energy of the orbit and μ the reduced mass.

Hint. Make use of the identity $\{f,gh\}=g\{f,h\}+\{f,g\}h$ and exploit the Poisson brackets calculated in the lecture.

Solution. We rewrite the LRL vector as

$$\mathbf{A}=\frac{1}{\mu}\mathbf{p}\times\mathbf{L}-\frac{\mathbf{r}}{r}=\frac{1}{\mu}\left(p^2-\frac{\mu}{r}\right)\mathbf{r}-\frac{1}{\mu}(\mathbf{r}\cdot\mathbf{p})\mathbf{p}. \tag{S.14}$$

We can now start to calculate

$$\begin{aligned}
\mu^2\{A_1,A_2\}&=\left\{\left(p^2-\frac{\mu}{r}\right)r_1-\frac{1}{\mu}(\mathbf{r}\cdot\mathbf{p})p_1,\left(p^2-\frac{\mu}{r}\right)r_2-\frac{1}{\mu}(\mathbf{r}\cdot\mathbf{p})p_2\right\}=\\
&=\left\{\left(p^2-\frac{\mu}{r}\right)r_1,\left(p^2-\frac{\mu}{r}\right)r_2\right\}+\left\{(\mathbf{r}\cdot\mathbf{p})p_1,(\mathbf{r}\cdot\mathbf{p})p_2\right\}-\left\{\left(p^2-\frac{\mu}{r}\right)r_1,(\mathbf{r}\cdot\mathbf{p})p_2\right\}+\left\{\left(p^2-\frac{\mu}{r}\right)r_2,(\mathbf{r}\cdot\mathbf{p})p_1\right\}=\\
&=\left(p^2-\frac{\mu}{r}\right)\left[r_1\left\{\left(p^2-\frac{\mu}{r}\right),r_2\right\}+r_2\left\{r_1,\left(p^2-\frac{\mu}{r}\right)\right\}\right]+(\mathbf{r}\cdot\mathbf{p})\left[p_1\{(\mathbf{r}\cdot\mathbf{p}),p_2\}+p_2\{p_1,(\mathbf{r}\cdot\mathbf{p})\}\right]\\
&-r_1p_2\left\{\left(p^2-\frac{\mu}{r}\right),(\mathbf{r}\cdot\mathbf{p})\right\}-\left(p^2-\frac{\mu}{r}\right)p_2\left\{r_1,(\mathbf{r}\cdot\mathbf{p})\right\}-(\mathbf{r}\cdot\mathbf{p})r_1\left\{\left(p^2-\frac{\mu}{r}\right),p_2\right\}\\
&-p_1r_2\left\{(\mathbf{r}\cdot\mathbf{p}),\left(p^2-\frac{\mu}{r}\right)\right\}-\left(p^2-\frac{\mu}{r}\right)p_1\left\{(\mathbf{r}\cdot\mathbf{p}),r_2\right\}-(\mathbf{r}\cdot\mathbf{p})r_2\left\{p_1,\left(p^2-\frac{\mu}{r}\right)\right\}=\\
&=\left(p^2-\frac{\mu}{r}\right)\left[r_1\left\{p^2,r_2\right\}+r_2\left\{r_1,p^2\right\}\right]+(\mathbf{r}\cdot\mathbf{p})[p_1p_2-p_2p_1]+q_1p_2\left(2p^2-\frac{\mu}{r}\right)-\left(p^2-\frac{\mu}{r}\right)p_2r_1\\
&+(\mathbf{r}\cdot\mathbf{p})r_1\left\{\frac{\mu}{r},p_2\right\}-p_1r_2\left(2p^2-\frac{\mu}{r}\right)+\left(p^2-\frac{\mu}{r}\right)p_1r_2+(\mathbf{r}\cdot\mathbf{p})r_2\left\{p_1,\frac{\mu}{r}\right\}=\\
&=\left(p^2-\frac{\mu}{r}\right)\left[-2r_1p_2+2r_2p_1-p_2r_1+r_2p_1\right]+\left(2p^2-\frac{\mu}{r}\right)\left[r_1p_2-p_1r_2\right]-(\mathbf{r}\cdot\mathbf{p})r_1\frac{\mu r_2}{r^3}+(\mathbf{r}\cdot\mathbf{p})p_1\frac{\mu r_2}{r^3}=\\
&=-\left(p^2-2\frac{\mu}{r}\right)\left[r_1p_2-r_2p_1\right]=-2EL_3.
\end{aligned} \tag{S.15}$$

The other two brackets follow from a permutation of the indices.