Exercise 1. ***The Motion of Vortices***

Consider a single two-dimensional vortex of strength ω placed in the origin. It produces a 2D velocity field given by

$$\mathbf{u}(\mathbf{r}) = \frac{\omega}{r}\mathbf{e}_\theta,$$

which can be rewritten in components as

$$u_x = -\frac{\omega y}{r^2},\ u_y = \frac{\omega x}{r^2}.$$

In this problem we will study the motion of many such vortices in their own velocity field. Suppose that we have n vortices of the same strength ω at distinct positions $\mathbf{r}_i = (x_i, y_i)$.

(a) Explain why the velocity of the i^{th} vortex is given by

$$\dot{x}_i = -\omega \sum_{j\neq i} \frac{y_i - y_j}{(x_i - x_j)^2 + (y_i - y_j)^2},\ \dot{y}_i = \omega \sum_{j\neq i} \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{1}$$

Quite surprisingly, this system can be described by using the Hamiltonian formalism. For each vortex we now define a generalized coordinate $q_i = x_i$ and identify $p_i = y_i$ with its 'canonical momentum'.

(b) Show that the equations (1) are equivalent to the Hamilton's equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k},\ \dot{p}_k = -\frac{\partial H}{\partial q_k} \text{ with } H = -\frac{\omega}{4}\sum_{\substack{i,j\\ i\neq j}} \log\left[(q_i - q_j)^2 + (p_i - p_j)^2\right]. \tag{2}$$

Hint: After differentiating H it is convenient to write the sum as

$$\frac{\partial H}{\partial q_k} = \sum_{j\neq i}\sum_{i} A_{ij}\delta_{ik} + \sum_{i\neq j}\sum_{j} B_{ij}\delta_{jk}$$

for some objects A_{ij}, B_{ij} and use the properties of the Kronecker δ_{ij}.

(c) Consider an infinitesimal rotation of the system given by

$$\delta q_i = -\epsilon p_i,\ \delta p_i = \epsilon q_i,\ i = 1, 2, ..., n \tag{3}$$

and show that it does not change the Hamiltonian to the order $\mathcal{O}(\epsilon)$. What is the generating function $\tilde{G}(q_1, ..., q_n, p_1, ..., p_n)$ of such transformation?

Hint: You should find the expansion $\log(1 + \epsilon^2) = \epsilon^2 + \mathcal{O}(\epsilon^4)$ useful. Recall that the generating function is defined by $\delta p_i = -\epsilon\partial\tilde{G}/\partial q_i$, $\delta q_i = \epsilon\partial\tilde{G}/\partial p_i$.

(d)* Define new coordinates $Q_i(\epsilon) = q_i + \delta q_i$, $P_i(\epsilon) = p_i + \delta p_i$ and the transformed Hamiltonian $H(\epsilon) = H(Q_i(\epsilon), P_i(\epsilon))$. Use the result from c) to deduce the value of $\mathrm{d}H/\mathrm{d}\epsilon|_{\epsilon=0}$. Hence show that $\tilde{G}$ is a conserved quantity, i.e. $\mathrm{d}\tilde{G}/\mathrm{d}t = [\tilde{G}, H] = 0$.

Solution.

(a) The velocity field at (x_i, y_i) due to the j^{th} vortex is described by

$$u_x = -\frac{\omega(y_i - y_j)}{|\mathbf{r}_i - \mathbf{r}_j|},\ u_y = \frac{\omega(x_i - x_j)}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

Summing contributions from all vortices $j \neq i$ gives the equations of motion.

(b) First we split the sum as suggested in the hint and use the sampling property of the Kronecker δ

$$\frac{\partial H}{\partial q_k} = -\frac{\omega}{4}\sum_{\substack{i,j\\ i\neq j}} \frac{2(q_i-q_j)(\delta_{ik}-\delta_{jk})}{(q_i-q_j)^2+(p_i-p_j)^2} = -\frac{\omega}{2}\sum_{j\neq i}\sum_{i} \frac{q_i-q_j}{(q_i-q_j)^2+(p_i-p_j)^2}\delta_{ik} + \frac{\omega}{2}\sum_{i\neq j}\sum_{j} \frac{q_i-q_j}{(q_i-q_j)^2+(p_i-p_j)^2}\delta_{jk} =$$

$$= -\frac{\omega}{2}\sum_{j\neq k} \frac{q_k-q_j}{(q_k-q_j)^2+(p_k-p_j)^2} + \frac{\omega}{2}\sum_{i\neq k} \frac{q_i-q_k}{(q_i-q_k)^2+(p_i-p_k)^2}$$

Now we relabell $i \to j$ in the second sum and see that the result can be written as

$$\frac{\partial H}{\partial q_k} = -\omega \sum_{j\neq k} \frac{q_k-q_j}{(q_k-q_j)^2+(p_k-p_j)^2} = \dot{y}_k.$$

The calculation for $\partial H/\partial p_k$ is analogous.

(c) The argument of the logarithm becomes after an ICT (3)

$$(q_i-\epsilon p_i - q_j + \epsilon p_j)^2 + (p_i + \epsilon q_i - p_j - \epsilon q_j)^2 = (q_i-q_j)^2 + (p_i-p_j)^2 + \left[(p_i-p_j)^2 + (q_i-q_j)^2\right]\epsilon^2$$

. (note that contributions of $\mathcal{O}(\epsilon)$ cancel after expanding the brackets). Hence we see that the transformed Hamiltonian is

$$H(\epsilon) = -\frac{\omega}{4}\sum_{\substack{i,j\\ i\neq j}} \log\left[\left((p_i-p_j)^2+(q_i-q_j)^2\right)(1+\epsilon^2)\right] = H(0) - \frac{\omega}{4}\sum_{\substack{i,j\\ i\neq j}}\log\left[1+\epsilon^2\right] = H(0) + \mathcal{O}(\epsilon^2)$$

and there is no term linear in ϵ.

The condition $\delta p_i = -\epsilon\partial\tilde{G}/\partial q_i$ implies $q_i = -\frac{\partial \tilde{G}}{\partial q_i}$. Similarly $p_i = -\partial\tilde{G}/\partial p_i$. These conditions are satisfied simultaneously by

$$\tilde{G} = -\frac{1}{2}\sum_i q_i^2 + p_i^2.$$

(d) $H(\epsilon) = H(0) + \mathcal{O}(\epsilon^2)$ implies that $\mathrm{d}H(\epsilon)/\mathrm{d}\epsilon|_{\epsilon=0} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[H(\epsilon)-H(0)] = 0$. But at the same time

$$\left.\frac{\mathrm{d}H}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = \left.\frac{\partial H}{\partial Q_i}\frac{\mathrm{d}Q_i}{d\epsilon}\right|_{\epsilon=0} + \left.\frac{\partial H}{\partial P_i}\frac{\mathrm{d}P_i}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = -\frac{\partial H}{\partial q_i}\frac{\partial\tilde{G}}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial\tilde{G}}{\partial q_i} = [\tilde{G}, H].$$

Thus $[\tilde{G}, H] = 0$.

Exercise 2. *One more exercise about Canonical transformations...*

Given the transformation for a system of one degree of freedom

$$\begin{aligned} Q &= q\cos\alpha - p\sin\alpha \\ P &= q\sin\alpha + p\cos\alpha \end{aligned} \tag{4}$$

(a) show that it satisfies the symplectic condition for any value of the parameter α.

(b) Find a generating function F for the transformation.
Hint: you can, for example, chose a generating function which depends on Q and q, so that $p = p(Q,q)$ and $P = P(Q,q)$, and use the transformation equations:

$$p = \frac{\partial F}{\partial q} \quad \text{and} \quad P = -\frac{\partial F}{\partial Q}\,. \tag{5}$$

(c) What is the physical significance of the transformation for $\alpha = 0$? For $\alpha = \pi/2$? Does your generating function work for both of these cases?

(d) * If not, can you find a generating function valid for the case where the one you have just found doesn't work?
Hint: try with a generating function which depends on other variables, e.g. q and P.

Solution.

(a) The symplectic condition is met if given the jacobian M is a symplettic matrix, i.e. it satisfies relation $M\,J\,M^T = J$. In our case

$$M = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Now we can calculate

$$\begin{aligned} MJM^T &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -\sin\alpha & \cos\alpha \\ -\cos\alpha & -\sin\alpha \end{pmatrix} \\ &= \begin{pmatrix} -\sin\alpha\,\cos\alpha + \sin\alpha\,\cos\alpha & \cos^2\alpha + \sin^2\alpha \\ -\cos^2\alpha - \sin^2\alpha & \sin\alpha\,\cos\alpha - \sin\alpha\,\cos\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J. \end{aligned}$$

(b) To find the generating function, we have to first make the choice on the variable F depends on. Given Eq. (5), it is natural to look for dependence on Q and q. Rearranging the first of Eq. (4) to solve it for $p(Q,q)$, we have

$$p(Q,q) = -\frac{Q}{\sin\alpha} + \frac{q\,\cos\alpha}{\sin\alpha} \tag{S.1}$$

then, integrating, we have

$$p = \frac{\partial F}{\partial q} \quad\Rightarrow\quad F = \int dq\, p(Q,q) \quad\Rightarrow\quad F = -\frac{Qq}{\sin\alpha} + \frac{q^2\,\cos\alpha}{2\sin\alpha} + g(Q) \tag{S.2}$$

After solving the second transformation of Eq. (4) for $P(Q,q)$, also using Eq. (S.1) to get rid of the p dependence, we get

$$P(Q,q) = q\sin\alpha - \frac{Q\cos\alpha}{\sin\alpha} + \frac{q\cos^2\alpha}{\sin\alpha} \tag{S.3}$$

which allow us to integrate the second of Eq. (5)):

$$\begin{aligned} P = -\frac{\partial F}{\partial Q} + h(q) \quad\Rightarrow\quad F &= -\int dQ\, P(Q,q) \\ &= -qQ\sin\alpha + \frac{1}{2}Q^2\cot\alpha - qQ\left(\frac{1}{\sin\alpha} - \sin\alpha\right) + h(q) \\ &= \frac{1}{2}Q^2\cot\alpha - \frac{qQ}{\sin\alpha} + h(q) \end{aligned} \tag{S.4}$$

Combining Eqs. (S.2) and (S.4) we get

$$F = \frac{1}{2}(q^2 + Q^2)\cot\alpha - \frac{qQ}{\sin\alpha} + c \tag{S.5}$$

(c) The transformation with $\alpha = 0$ is simply the identy transformation $P = p$, $Q = q$. for $\alpha = \pi/2$ is an exchange of variables $P = q$, $Q = -p$.

This particular choice of generating function, is not defined for $\alpha = 0$, but it is not problematic for $\alpha = \pi/2$.

(d) We could chose a generating function which depends on another combination of variables, for example P and q, and the transformations equations would be different from Eq. (5), i.e.

$$p = \frac{\partial F}{\partial q} \quad \text{and} \quad Q = \frac{\partial F}{\partial P} . \tag{S.6}$$

and found a different generating function. In this case we need to rewrite the second transformation of Eq. (4) as

$$p = \frac{P}{\cos\alpha} - q\,\frac{\sin\alpha}{\cos\alpha} , \tag{S.7}$$

which can be used to integrate the first of Eq. (S.6):

$$F = \int dq\, p(q,P) + h(P) = \frac{q\,P}{\cos\alpha} - \frac{q^2}{2}\,\tan\alpha + h(P) \tag{S.8}$$

We also have that

$$\begin{aligned} Q &= q\,\cos\alpha - p\,\sin\alpha = q\,\cos\alpha - \left(\frac{P}{\cos\alpha} - q\,\frac{\sin\alpha}{\cos\alpha}\right)\sin\alpha \\ &= q\,\frac{\cos^2\alpha + \sin^2\alpha}{\cos\alpha} - P\,\tan\alpha = \frac{q}{\cos\alpha} - P\,\tan\alpha \end{aligned} \tag{S.9}$$

which leads to

$$F = \int dP\, Q(q,P) + g(q) = \frac{q\,P}{\cos\alpha} - \frac{P^2}{2}\,\tan\alpha + g(q) . \tag{S.10}$$

Combining Eq. (S.8) with Eq. (S.10) we get

$$F = \int dP\, Q(q,P) + g(q) = \frac{q\,P}{\cos\alpha} - \frac{1}{2}\,(P^2 + q^2)\tan\alpha + c , \tag{S.11}$$

which is singular for $\alpha = \pi/2$ but is fine for $\alpha = 0$.

Exercise 3. *Conserved tensor for harmonic oscillator*

We have seen that the Kepler problem features, in addition to the total energy and angular momentum, a conserved quantity known as Laplace-Runge-Lenz vector. It turns out that the existence of conserved Laplace-Runge-Lenz-like vectors are not specific to the Kepler potential. Instead, it is rather a feature of radial force problems.

As such it is not surprising that also the three-dimensional isotropic harmonic oscillator allows for such a conserved quantity, which in this case is the (six component) tensor

$$\mathbf{A}_{ij} = \frac{1}{\lambda}(p_i p_j + \lambda^2 r_i r_j) , \tag{6}$$

where $i, j \in \{x, y, z\}$.

(a) Consider a particle of mass m in the harmonic potential

$$V(\vec{r}) = \frac{1}{2} u \vec{r}^{\,2} , \qquad \vec{r} \in \mathbb{R}^3. \tag{7}$$

Write down the Lagrangian and find the Hamiltonian from the Legendre transformation.

Solution. The kinetic energy is given by

$$T = \frac{1}{2} m \dot{\vec{r}}^2$$

the potential energy according to Eq. (7). This results in the Lagrangian

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - \frac{1}{2} u \vec{r}^{\,2} .$$

To find the Hamiltonian, we introduce the canonical momenta

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = m \dot{r}_i$$

and obtain

$$H = \sum_i p_i \dot{q}_i - L = \frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) + \frac{1}{2} u ({r_x}^2 + {r_y}^2 + {r_z}^2) .$$

(b) Using Poisson brackets $[\cdot, \cdot]$ and a suitable choice of λ, show that the tensor $\mathbf{A}_{ij}$ defined in Eq. (6) is conserved, i.e.,

$$[\mathbf{A}_{ij}, H] = 0 , \qquad \forall\, i, j . \tag{8}$$

Hint. Although the problem has radial symmetry, note that it might be simpler to carry out the calculation in cartesian coordinates.

Solution. Using the definition of the Poisson bracket we find

$$[\mathbf{A}_{ij}, H] = \sum_k \left(\frac{\partial A_{ij}}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial A_{ij}}{\partial p_k} \frac{\partial H}{\partial q_k} \right) .$$

We consider therefore

$$\frac{\partial A_{ij}}{\partial q_k} = \frac{\partial A_{ij}}{\partial r_k} = \lambda(\delta_{ik} r_j + \delta_{jk} r_i) , \qquad \frac{\partial A_{ij}}{\partial p_k} = \frac{1}{\lambda}(\delta_{ik} p_j + \delta_{jk} p_i) ,$$

and

$$\frac{\partial H}{\partial q_k} = \frac{\partial H}{\partial r_k} = u r_k , \qquad \frac{\partial H}{\partial p_k} = \frac{1}{m} p_k .$$

Putting everything together, we obtain

$$[\mathbf{A}_{ij}, H] = \sum_k \left(\frac{\lambda}{m}(\delta_{ik} r_j + \delta_{jk} r_i) p_k - \frac{u}{\lambda}(\delta_{ik} p_j + \delta_{jk} p_i) r_k \right) = \frac{\lambda}{m}(r_j p_i + r_i p_j) - \frac{u}{\lambda}(p_j r_i + p_i r_j) ,$$

$$= (r_j p_i + r_i p_j) \frac{\lambda}{m} \left(1 - \frac{um}{\lambda^2} \right) = 0 \qquad \Leftrightarrow \qquad \lambda^2 = um .$$

(c) The tensor $\mathbf{A}_{ij}$ has six components, corresponding to six constants of motion. The energy and the angular momentum are additional constants of motion, because the harmonic oscillator is a central force problem. Hence there are more constants of motion than degrees of freedom. For this reason some of them must be interlinked.

In this context, relate

$$\operatorname{tr} \mathbf{A}_{ij} = \sum_i \mathbf{A}_{ii} \tag{9}$$

to another conserved quantity.

Solution. Using the definition of $\mathbf{A}_{ij}$ we find

$$\operatorname{tr} \mathbf{A}_{ij} = \sum_i \mathbf{A}_{ii} = \sum_i \frac{1}{\lambda}(p_i p_i + \lambda^2 r_i r_i) = \sqrt{\frac{m}{u}} \sum_i \left(\frac{1}{m} p_i p_i + u r_i r_i \right) = 2 \sqrt{\frac{m}{u}} E ,$$

where in the last step we used that the energy is conserved and equal to E.