Exercise 1. ***Routh's Procedure***

In this exercise you will learn to apply the Routh's procedure to a central potential problem.

Suppose we have a system of n degrees of freedom, and $n-s$ coordinates are cyclic. We label them as $q_{s+1}, \cdots, q_n$. Then we introduce the Routhian:

$$R(q_1, \cdots, q_n; \dot{q}_1, \cdots, \dot{q}_s; p_{s+1}, \cdots, p_n; t) = \sum_{i=s+1}^{n} p_i \dot{q}_i - \mathcal{L} \tag{1}$$

$$= H_{\text{cycl}}(p_{s+1}, \cdots, p_n) - \mathcal{L}_{\text{noncycl}}(q_1, \cdots, q_s; \dot{q}_1, \cdots, \dot{q}_s) \tag{2}$$

Then it is apparent that

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i} = 0 \qquad i = 1, \cdots, s \tag{3}$$

$$\frac{\partial R}{\partial p_i} = \dot{q}_i, \quad \frac{\partial R}{\partial q_i} = -\dot{p}_i = 0, \qquad i = s+1, \cdots, n \tag{4}$$

so the Routhian is Hamiltonian on the cyclic variables and Lagrangian on the non-cyclic ones.

In order to understand the Routh's procedure better, consider the Lagrangian for central potential:

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{5}$$

where

$$U(r) = -\frac{k}{r^n} \tag{6}$$

a) Determine the cyclic variable and write down the Routhian.

Solution. θ is the cyclic variable.

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} : \text{constant} \tag{S.1}$$

$$R(r; \dot{r}; p_\theta) = \frac{p_\theta^2}{2mr^2} - \frac{m\dot{r}^2}{2} - \frac{k}{r^n} \tag{S.2}$$

so the Routhian is the effective potential minus the kinetic energy.

b) Apply the Euler-Lagrange equations (3) to the noncyclic coordinate to obtain the equation of motion.

Solution. E-L equation on the non-cyclic variable is:

$$m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{kn}{r^{n+1}} = 0 \tag{S.3}$$

c) Apply now the Hamilton's equation (4) to the cyclic variable.

Solution. Hamilton's equation on the cyclic variable:

$$\frac{\partial R}{\partial \theta} = -\dot{p}_\theta = 0, \qquad \frac{\partial R}{\partial p_\theta} = \frac{p_\theta}{mr^2} = \dot{\theta} \tag{S.4}$$

which is the same solution we got before: angular momentum is conserved:

$$p_\theta = mr^2\dot{\theta} : \text{constant} \tag{S.5}$$

Routh's procedure does not add any physical content; it just streamlines the problem. It is convenient for problems that have many variables.

Exercise 2. *Canonical Transformations*

The transformation equations between two sets of coordinates are

$$Q = \log(1 + q^{1/2}\cos p) \tag{7}$$
$$P = 2(1 + q^{1/2}\cos p)q^{1/2}\sin p. \tag{8}$$

a) Using the symplectic criterion, show from these transformation equations that Q, P are canonical variables if q and p are.

b) Show the same thing using Poisson-brackets.

c) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p.$$

Solution.

a) We find the Jacobian

$$\begin{aligned} M &= \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\frac{q^{-1/2}\cos p}{1+q^{1/2}\cos p} & -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + 2\cos p \sin p & 2q^{1/2}\cos p + 2\cos^2 p - 2q\sin^2 p \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\frac{q^{1/2}\cos p}{1+q^{1/2}\cos p} & -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \\ q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix}. \end{aligned}$$

Now we can calculate

$$\begin{aligned} M^T J M &= \begin{pmatrix} \frac{1}{2}\frac{q^{1/2}\cos p}{1+q^{1/2}\cos p} & q^{-1/2}\sin p + \sin 2p \\ -\frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} & 2q^{1/2}\cos p + 2q\cos 2p \end{pmatrix} \\ &\times \begin{pmatrix} q^{-1/2}\sin p + \sin 2p & 2q^{1/2}\cos p + 2q\cos 2p \\ -\frac{1}{2}\frac{q^{1/2}\cos p}{1+q^{1/2}\cos p} & \frac{q^{1/2}\sin p}{1+q^{1/2}\cos p} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{\cos^2 p + \sin^2 p + q^{1/2}\cos p\cos 2p + q^{1/2}\sin p \sin 2p}{1+q^{1/2}\cos p} \\ -\frac{\cos^2 p + \sin^2 p + q^{1/2}\cos p\cos 2p + q^{1/2}\sin p \sin 2p}{1+q^{1/2}\cos p} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= J. \end{aligned}$$

b) We have that $[Q,Q]=0$, $[P,P]=0$ and

$$\begin{aligned}
[Q,P] &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= \frac{q^{-\frac{1}{2}}\cos p}{1+q^{\frac{1}{2}}\cos p}\left(q^{\frac{1}{2}}\cos p + q\cos^2 p - q\sin^2 p\right) \\
&\quad + \frac{q^{\frac{1}{2}}\sin p}{1+q^{\frac{1}{2}}\cos p}\left(q^{-\frac{1}{2}}\sin p + 2\cos p \sin p\right) \\
&= \frac{q^{\frac{1}{2}}\cos p \sin^2 p + q^{\frac{1}{2}}\cos^3 p + 1}{1+q^{\frac{1}{2}}\cos p} \\
&= 1.
\end{aligned}$$

Therefore the transformation is canonical. Then if q and p are canonical variables, so are Q and P.

c) We need to varify that for the given F_3

$$q = -\frac{\partial F_3}{\partial p},\, P = -\frac{\partial F_3}{\partial Q}.$$

Now

$$-\frac{\partial F_3}{\partial p} = (e^Q - 1)^2 \frac{1}{\cos^2 p} \tag{S.6}$$

$$-\frac{\partial F_3}{\partial Q} = 2e^Q(e^Q-1)\tan p. \tag{S.7}$$

Setting $q = -\frac{\partial F_3}{\partial p}, P = -\frac{\partial F_3}{\partial Q}$ and rewriting for Q and P in terms of q and p yields

$$Q = \log(1 + q^{1/2}\cos p) \tag{S.8}$$

$$P = 2q^{1/2}\sin p + q\sin 2p \tag{S.9}$$

as required.

Exercise 3. *Canonical transformations with two coordinates*

Consider the transformation

$$\begin{cases} Q_1 = q_1, \\ Q_2 = p_2, \end{cases} \qquad \begin{cases} P_1 = p_1 - 2p_2, \\ P_2 = -2q_1 - q_2. \end{cases} \tag{9}$$

Prove that such a transformation is canonical
a) using the symplectic criterion,
b) using Poisson brackets.

Solution.

a) Defining $\vec{\omega}^T = (\vec{q}, \vec{p})$, $\vec{\Omega}^T = (\vec{Q}, \vec{P})$ we have

$$\mathbf{M} = \frac{\partial \Omega_i}{\partial \omega_j} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{array}\right). \tag{S.10}$$

Thus we find

$$\mathbf{MJM}^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ -2 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \mathbf{J}, \tag{S.11}$$

and the criterion is fulfilled.

b) Assuming canonical Poisson brackets for the initial coordinates,

$$[q_1, q_2] = 0, \qquad [p_1, p_2] = 0, \qquad [q_1, p_2] = 0, \qquad [q_2, p_1] = 0, \qquad [q_1, p_1] = 1, \qquad [q_2, p_2] = 1; \tag{S.12}$$

and recalling that Poisson brackets inherit linearity from derivatives, we find

$$\begin{aligned}
[Q_1, Q_2] &= [q_1, p_2] = 0, \\
[P_1, P_2] &= [p_1 - 2p_2, -2q_1 - q_2] = [p_1, -2q_1 - q_2] - 2[p_2, -2q_1 - q_2] = -2[p_1, q_1] + 2[p_2, q_2] = 0, \\
[Q_1, P_2] &= [q_1, -2q_1 - q_2] = 0, \\
[Q_2, P_1] &= [p_2, p_1 - 2p_2] = 0, \\
[Q_1, P_1] &= [q_1, p_1 - 2p_2] = [q_1, p_1] - 2[q_1, p_2] = 1, \\
[Q_2, P_2] &= [p_2, -2q_1 - q_2] = -2[p_2, q_1] - [p_2, q_2] = [q_2, p_2] = 1.
\end{aligned} \tag{S.13}$$

Note that all brackets are meant with respect to the original variables $(\vec{q}, \vec{p})$ and that antisymmetry has been used throughout. Also, brackets of a variable with itself have not been considered because they vanish trivially. We explicitly checked that canonical Poisson brackets are preserved by the transformation, which guarantees that the transformation itself is canonical and concludes the exercise.

Exercise 4. *Hamiltonian with dissipative force*

A particle of mass m moves in one dimension q in a potential $V(q)$ and is subject to a damping force $F = -2m\gamma\dot{q}$ proportional to its velocity.

a) Show that the equation of motion can be obtained from the lagrangian

$$L[q, \dot{q}, t] = e^{2\gamma t}\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]. \tag{10}$$

b) Compute the canonical momentum p conjugate to q and find the Hamiltonian $H[q, p, t]$.

c) Using the generating function

$$F_2(q, P, t) = qPe^{\gamma t}, \tag{11}$$

find the transformed Hamiltonian $K[Q, P, t]$.

Solution.

a) The Euler-Lagrange equation reads

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = -\frac{\partial V}{\partial q}e^{2\gamma t} - \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{2\gamma t}m\dot{q}\right] = -e^{2\gamma t}\left[m\ddot{q} + 2m\gamma\dot{q} + \frac{\partial V}{\partial q}\right] \overset{!}{=} 0, \tag{S.14}$$

or

$$m\ddot{q} = -\frac{\partial V}{\partial q} - 2m\gamma\dot{q}. \tag{S.15}$$

This is indeed Newton's equation for a particle in a potential V and with the given damping force.

b) The canonical momentum can be immediately obtained by its definition

$$p = \frac{\partial L}{\partial \dot{q}} = e^{2\gamma t}m\dot{q}. \tag{S.16}$$

Conversely, $\dot{q} = pe^{-2\gamma t}/m$ is the substitution needed to eliminate the variable $\dot{q}$. The hamiltonian is thus

$$H = p\dot{q} - L = p\dot{q} - \left[\frac{1}{2}p\dot{q} - e^{2\gamma t}V(q)\right] = \frac{p^2}{2m}e^{-2\gamma t} + V(q)e^{2\gamma t}. \tag{S.17}$$

c) The given generating function yields

$$Q = \frac{\partial}{\partial P} F_2(q, P, t) = q e^{\gamma t}, \qquad p = \frac{\partial}{\partial q} F_2(q, P, t) = P e^{\gamma t}, \tag{S.18}$$

which results in

$$K[Q, P, t] = H + \frac{\partial F_2}{\partial t} = \frac{P^2}{2m} + V(Qe^{-\gamma t})e^{2\gamma t} + \gamma QP. \tag{S.19}$$

Now consider a harmonic oscillator of potential

$$V(q) = \frac{1}{2} m\omega^2 q^2. \tag{12}$$

d) Which of the Hamiltonians H and K is a constant of motion? Why?
e) In the underdamped case $\gamma < \omega$, obtain the solution $q(t)$ and express the integration constant related to the oscillation amplitude in terms of the conserved quantity.

Solution.

d) For the given potential the two hamiltonians read

$$H[q, p, t] = \frac{p^2}{2m} e^{-2\gamma t} + \frac{1}{2} m\omega^2 q^2 e^{2\gamma t}, \qquad K[Q, P] = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 Q^2 + \gamma QP. \tag{S.20}$$

For a generic Hamiltonian $H[q, p, t]$, using the chain rule one has

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}. \tag{S.21}$$

Using the equations of motion one then finds

$$\frac{\mathrm{d}H}{\mathrm{d}t} = [H, H] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}. \tag{S.22}$$

Given that K has no explicit time dependence it is a conserved quantity.

e) Hamilton's equations read

$$\begin{cases} \dot{Q} = \dfrac{\partial K}{\partial P} = \dfrac{P}{m} + \gamma Q, \\ \dot{P} = -\dfrac{\partial K}{\partial Q} = -m\omega^2 Q - \gamma P, \end{cases} \tag{S.23}$$

thus

$$\ddot{Q} = \frac{\dot{P}}{m} + \gamma\dot{Q} = -(\omega^2 - \gamma^2)Q. \tag{S.24}$$

Setting $\tilde{\omega}^2 = \omega^2 - \gamma^2$ one therefore finds

$$Q(t) = Q_0 \cos(\tilde{\omega}t + \phi), \qquad \Rightarrow \qquad q(t) = Q_0 e^{-\gamma t} \cos(\tilde{\omega}t + \phi). \tag{S.25}$$

Using the first of Hamilton's equation

$$P = m(\dot{Q} - \gamma Q), \tag{S.26}$$

we find

$$K = \frac{m}{2}[(\dot{Q} - \gamma Q)^2 + \omega^2 Q^2 + 2\gamma Q(\dot{Q} - \gamma Q)] = \frac{m}{2}[\dot{Q}^2 + (\omega^2 - \gamma^2)Q^2] = \frac{m}{2}\tilde{\omega}^2 Q_0^2. \tag{S.27}$$

This gives

$$Q_0 = \sqrt{\frac{2K}{m\tilde{\omega}^2}}. \tag{S.28}$$