Exercise 1. ***Bar supported by springs***

A uniform horizontal thin bar of length L and mass M is supported at its ends by two vertical springs with force constants k_1 and k_2 respectively (see figure). At equilibrium, the springs have the same length and the bar lies horizontally. At $t = 0$ the bar is displaced from its equilibrium position and let free to move.

(a) Derive an expression for the moment of inertia of the bar about its center of mass.

(b) Choose a suitable set of generalised coordinates and find an expression for the Lagrangian.

(c) Linearise the system under the assumption of small oscillations and find the equations of motion. Assume that the center of mass of the bar stays on the same vertical line.

Hint. You can either start from the Lagrangian calculated in the previous question and Taylor expand it, assuming the motion around the equilibrium is small, or you can reformulate a new Lagrangian for the simplified problem and derive the equations of motions from it.

(d) Find the eigenmodes of the oscillations for the general case $k_1 \neq k_2$ and for the special case $k_1 = k_2 = k$.

Solution.

(a) Let x be the axis parallel to the long edge of the bar and η an axis of symmetry going through the center of mass of the bar, perpendicularly to its long edge (because of the symmetry of the rod it will not matter which axis we choose). The moment of inertia of the bar about this axis is given by integrating over the full spatial extent the differential mass $\mathrm{d}m(x)$ multiplied by the square of the distance to the axis:

$$I_\eta = \int_{-L/2}^{L/2} x^2 \mathrm{d}m. \tag{S.1}$$

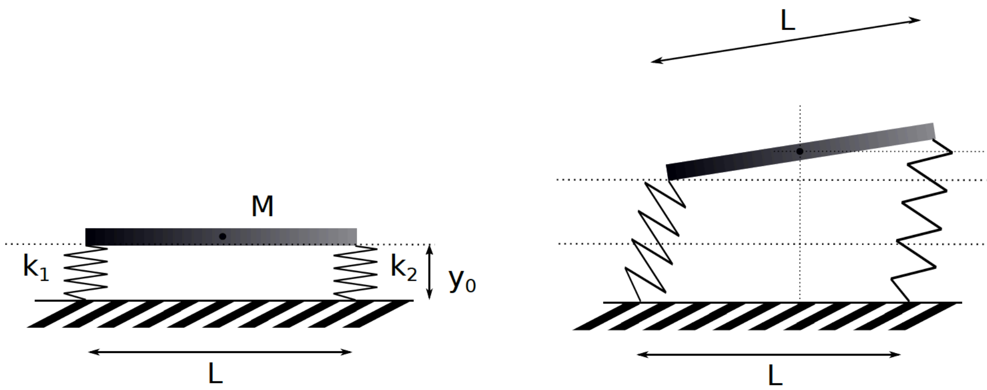

Figure 1: Sketch for the problem of a bar resting on two springs.

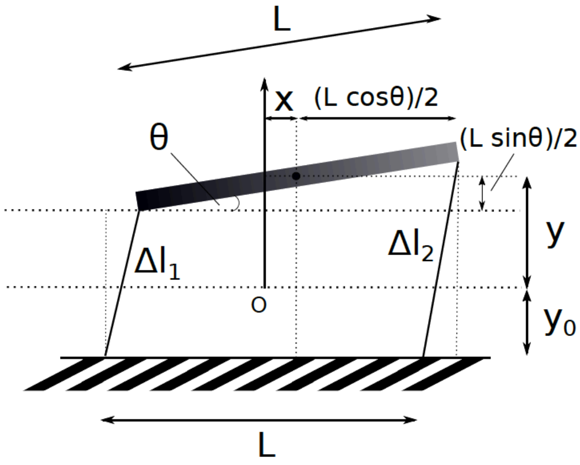

Figure 2: Generalised coordinates for the problem of a bar supported by two springs.

Since the bar is homogeneous, the differential mass is simply given by $\mathrm{d}m = \lambda \mathrm{d}x$, with $\lambda = M/L$ being the mass density per unit length. Performing the integration we obtain:

$$I_\eta = \lambda \int_{-L/2}^{L/2} x^2 \mathrm{d}x \tag{S.2}$$

$$= \lambda \frac{x^3}{3}\Big|_{-L/2}^{L/2} \tag{S.3}$$

$$= \frac{M}{L}\left(\frac{L^3}{24} - \frac{-L^3}{24}\right) \tag{S.4}$$

$$= \frac{ML^2}{12}. \tag{S.5}$$

(b) We can introduce a coordinate system with the origin lying in the center of mass of the bar, the x-axis stretching horizontally to the right and the y-axis pointing upward perpendicularly to the bar. When the bar is pulled away from equilibrium, the center of mass will generally be shifted to another position, which we can parametrise by the coordinates (x, y). Additionally, the bar will not necessarily lie horizontally anymore, but its axis will form an angle θ with the horizontal x-axis. Therefore, a suitable choice of generalised coordinates will be (x, y, θ) and the kinetic energy of the bar is readily given in terms of the their generalised velocities as:

$$T = T_T + T_R = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2, \tag{S.6}$$

where T_T is the translational kinetic energy of the center of mass and T_R is the rotational contribution around it. In the previous section we calculated the moment of inertia of the bar, which is $I = \frac{ML^2}{12}$. For the potential energy we need an expression for the lengths of the springs when they are stretched out of the equilibrium position. These lengths are given by Δl_1 and Δl_2 in the figure above and can be calculated from geometrical considerations as the distance between the left (right) end of the bar with coordinates $(\mp L/2, -y_0)$ and the left (right) supporting point with coordinates $(\mp \frac{L}{2}\cos\theta + x, y \mp \frac{L}{2}\sin\theta)$ as:

$$\Delta l_1 = \sqrt{\left(-\frac{L}{2}\cos\theta + x + \frac{L}{2}\right)^2 + \left(y_0 + y - \frac{L}{2}\sin\theta\right)^2} \tag{S.7}$$

and

$$\Delta l_2 = \sqrt{\left(\frac{L}{2}\cos\theta + x - \frac{L}{2}\right)^2 + \left(y_0 + y + \frac{L}{2}\sin\theta\right)^2}. \tag{S.8}$$

The potential energy of the system is the sum of the gravitational potential energy acting on the center of mass of the bar and the mechanical energy stored in both springs

$$U = U_G + U_{S_1} + U_{S_2} \tag{S.9}$$

and with the expressions for Δl_1 and Δl_2 is therefore computed as:

$$U = Mgy + \frac{1}{2}k_1(\Delta l_1 - y_0)^2 + \frac{1}{2}k_2(\Delta l_2 - y_0)^2 \tag{S.10}$$

$$= Mgy + \frac{1}{2}k_1\left(\sqrt{\left(-\frac{L}{2}\cos\theta + x + \frac{L}{2}\right)^2 + \left(y_0 + y - \frac{L}{2}\sin\theta\right)^2} - y_0\right)^2 + + \frac{1}{2}k_2\left(\sqrt{\left(\frac{L}{2}\cos\theta + x - \frac{L}{2}\right)^2 + \left(y_0 + y + \frac{L}{2}\sin\theta\right)^2} - y_0\right)^2. \tag{S.11}$$

The Lagrangian of the systems is therefore

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 - Mgy + - \frac{1}{2}k_1\left(\sqrt{\left(-\frac{L}{2}\cos\theta + x + \frac{L}{2}\right)^2 + \left(y_0 + y - \frac{L}{2}\sin\theta\right)^2} - y_0\right)^2 - \frac{1}{2}k_2\left(\sqrt{\left(\frac{L}{2}\cos\theta + x - \frac{L}{2}\right)^2 + \left(y_0 + y + \frac{L}{2}\sin\theta\right)^2} - y_0\right)^2 \tag{S.12}$$

(c) We see that the system's motion is very chaotic and the equations of motion cannot be solved analytically. Therefore we perform a linearisation procedure assuming that the angle θ is very small. In this regime $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. The Lagrangian is then simplified to:

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 - Mgy + - \frac{1}{2}k_1\left(\sqrt{x^2 + \left(y_0 + y - \frac{L}{2}\theta\right)^2} - y_0\right)^2 - \frac{1}{2}k_2\left(\sqrt{x^2 + \left(y_0 + y + \frac{L}{2}\theta\right)^2} - y_0\right)^2. \tag{S.13}$$

Assuming that the motion around equilibrium is small, we can further simplify the square root by expanding in a Taylor series:

$$\sqrt{x^2 + \left(y_0 + y \pm \frac{L}{2}\theta\right)^2} - y_0 = y_0\sqrt{1 + \frac{x^2 + y^2}{y_0^2} + \frac{L^2\theta^2}{4y_0^2} + \frac{2y}{y_0} \pm \frac{yL\theta}{y_0^2} \pm \frac{L\theta}{y_0}} - y_0 \tag{S.14}$$

$$\approx \frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y \pm \frac{yL\theta}{2y_0} \pm \frac{L\theta}{2} \tag{S.15}$$

With this expansion the Lagrangian assumes the following form:

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 - Mgy + - \frac{1}{2}k_1\left(\frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y - \frac{yL\theta}{2y_0} - \frac{L\theta}{2}\right)^2 - \frac{1}{2}k_2\left(\frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y + \frac{yL\theta}{2y_0} + \frac{L\theta}{2}\right)^2. \tag{S.16}$$

The equations of motions then become:

$$M\ddot{x} = -k_1\left(\frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y - \frac{yL\theta}{2y_0} - \frac{L\theta}{2}\right)\left(\frac{x}{y_0}\right) + - k_2\left(\frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y + \frac{yL\theta}{2y_0} + \frac{L\theta}{2}\right)\left(\frac{x}{y_0}\right) \tag{S.17}$$

$$M\ddot{y} = -Mg - k_1\left(\frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y - \frac{yL\theta}{2y_0} - \frac{L\theta}{2}\right)\left(\frac{y}{y_0} + 1 - \frac{L\theta}{2y_0}\right) + - k_2\left(\frac{x^2 + y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y + \frac{yL\theta}{2y_0} + \frac{L\theta}{2}\right)\left(\frac{y}{y_0} + 1 + \frac{L\theta}{2y_0}\right) \tag{S.18}$$

$$I\ddot{\theta} = -k_1\left(\frac{x^2+y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y - \frac{yL\theta}{2y_0} - \frac{L\theta}{2}\right)\left(\frac{L^2\theta}{4y_0} - \frac{yL}{2y_0} - \frac{L}{2}\right)$$
$$- k_2\left(\frac{x^2+y^2}{2y_0} + \frac{L^2\theta^2}{8y_0} + y + \frac{yL\theta}{2y_0} + \frac{L\theta}{2}\right)\left(\frac{L^2\theta}{4y_0} + \frac{yL}{2y_0} + \frac{L}{2}\right) \tag{S.19}$$

This is now a system of polynomial differential equations that can be solved numerically. To obtain an approximate analytical solution, we restrict our calculations only to the linear terms in x, y and θ, discarding terms $\sim x\theta$ and $\sim y\theta$. This is equivalent to neglecting contribution of the potential energy coming from an extension of the springs in the x-direction We also neglect gravity to obtain a homogeneous system of linear differential equations:

$$\begin{cases} \ddot{x} = 0 \\ \ddot{y} = -\frac{k_1+k_2}{M}y + \frac{L(k_1-k_2)}{2M}\theta \\ \ddot{\theta} = \frac{L(k_1-k_2)}{2I}y - \frac{L^2(k_1+k_2)}{4I}\theta \end{cases} \tag{S.20}$$

or in matrix notation:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(\frac{k_1+k_2}{M}) & \frac{L(k_1-k_2)}{2M} \\ 0 & \frac{L(k_1-k_2)}{2I} & -\frac{L^2(k_1+k_2)}{4I} \end{pmatrix} \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}. \tag{S.21}$$

Assuming that the motion of the springs was restricted to be along the vertical axis, we could directly have computed the simplified Lagrangian:

$$\mathcal{L} = \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 - Mgy - \frac{1}{2}k_1\left(y - \frac{L}{2}\sin\theta\right)^2 - \frac{1}{2}k_2\left(y + \frac{L}{2}\sin\theta\right)^2 \tag{S.22}$$

Upon linearisation $\sin\theta \approx \theta$, this Lagrangian leads to the same equations of motion.

(d) From equation (S.21) we see that in the limit of small oscillations, the eigenmode of the x-component is zero and therefore this coordinate decouples from the motion, which is given only in terms of y and θ. With $I = \frac{ML^2}{12}$ the matrix

$$\begin{pmatrix} -(\frac{k_1+k_2}{M}) & \frac{L(k_1-k_2)}{2M} \\ \frac{6(k_1-k_2)}{ML} & -\frac{3(k_1+k_2)}{M} \end{pmatrix} \tag{S.23}$$

has eigenfrequencies

$$\omega_1^2 = \frac{2}{M}\left[K + \Delta k\sqrt{1+\alpha}\right] \qquad \omega_2^2 = \frac{2}{M}\left[K - \Delta k\sqrt{1+\alpha}\right] \tag{S.24}$$

where

$$K \equiv k_1 + k_2, \qquad \Delta k \equiv k_1 - k_2, \qquad \alpha \equiv \frac{k_1 k_2}{(k_1-k_2)^2}. \tag{S.25}$$

The eigenvectors are:

$$v_1 = \begin{pmatrix} 1 \\ \frac{-2K}{L\Delta k} + \frac{4}{L}\sqrt{1+\alpha} \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ \frac{-2K}{L\Delta k} - \frac{4}{L}\sqrt{1+\alpha} \end{pmatrix}. \tag{S.26}$$

The eigenmodes correspond to a mixture of vertical displacement and circular motion around the center of mass, depending on the values of the spring constants. The rotational motion will be symmetric around a given angle determined by K and Δk. Note that the horizontal position x of the bar is unchanged! This is not the case anymore, if the springs are attached on different points along the bar. For $k_1 > k_2$, the spring on the left hand side will tend to be harder to compress and therefore the angle will be positive (tilting towards the left, compressing the left spring). For $k_1 < k_2$, the spring on the left hand will on the contrary be softer and the bar will prefer an orientation with negative angles (tilting towards the right, compressing the right spring).

When the spring constants of the two springs are equal — $k_1 = k_2 = k$ — the coefficient matrix is already diagonal:

$$\begin{pmatrix} -\frac{2k}{M} & 0 \\ 0 & -\frac{6k}{M} \end{pmatrix}. \tag{S.27}$$

In this regime the generalised coordinates are fully decoupled and the eigenmodes are represented by either a pure vertical oscillation of the horizontal bar, or an alternating circular motion around its center of mass with no vertical displacement.

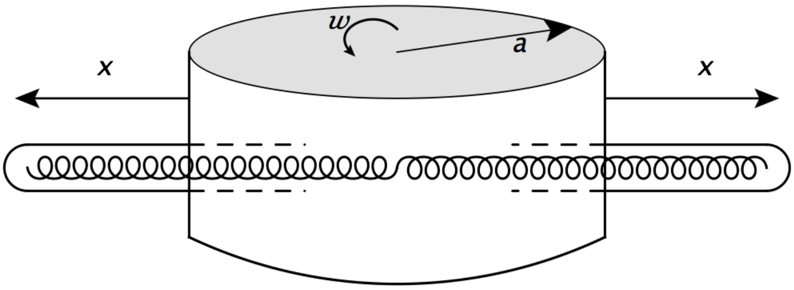

Figure 3: Sketch of the satellite with two antennae.

Exercise 2. ***Satellite with antennae***

A cylindrical satellite of mass M and radius a is spinning about its axis with initial angular velocity $\bar{\omega}$. It carries within it two hollow antennae, each of mass m and length $2\,a$ lying one within the other along a diameter (the mass of the antennae it taken into account by the total mass of the satellite M). At the beginning these antennae are fully within the satellite, but they start being forced out **symmetrically** in opposite directions at constant speed, stretching a spring of strength constant K and natural length $2\,a$ which joins their ends.

(a) Set up the Lagrangian for the system assuming the spring has negligible mass (this involves the computation of the moment of inertia of the system as a function the position of the antennae).
Hint: describe the position of the antennae through the variable x, which is zero when the antennae are fully inside the satellite and $2a$ when they are fully outside.

(b) By using angular momentum conservation in the absence of external forces, find that the angular velocity of the system when the antennae are fully outside the satellite is

$$\omega(x = 2a) = \frac{M}{M + 16\,m}\bar{\omega} \tag{1}$$

(c) Show that it is possible to choose K so that no net work is done by the motor driving the antennae in fully extending them a distance $2\,a$ out of the satellite. Find that

$$K = \frac{1}{2}\frac{M\,m}{M + 16\,m}\,\bar{\omega}^2\,. \tag{2}$$

Solution.

(a) The first thing that one should do in problems involving the rotation of rigid bodies, such as this, is the computation of the moment of inertial. Let us start considering the infinitesimal mass element of the cylindrical satellite

$$dm = \rho\,h\,2\pi r\,dr \tag{S.28}$$

with h being the height of the cylinder. The mass density ρ is

$$\rho = \frac{M}{\pi\,a^2\,h}\,. \tag{S.29}$$

The moment of inertia of the cylinder is therefore given by

$$I_M = \int dm\,r^2 = \rho\,h\,2\pi\int_0^a dr\,r^3 = \rho\,h\,2\pi\frac{a^4}{4} = \frac{1}{2}M\,a^2 \tag{S.30}$$

Let us now turn our attention to the antennae: it's linear mass density is given by

$$\frac{dm}{dr} = \frac{m}{2a}, \tag{S.31}$$

When $0 \le x \le a$ (i.e. when less then half of the antenna is outside the main body of the satellite, and therefore there are parts of the antennae on both sides of the axis of rotation) the moment of inertia gets two contribution:

$$I_{a_1} = \int_0^{a+x} dm\, r^2 + \int_0^{a-x} dm\, r^2 = \frac{m}{2\,a}\left[\int_0^{a+x} dr\, r^2 + \int_0^{a-x} dr\, r^2\right] = \frac{m}{2\,a}\left[\frac{(a+x)^3}{3} + \frac{(a-x)^3}{3}\right] \tag{S.32}$$

the first being the contribution on one side of the rotation axis of the satellite, the other been on the opposite side. When $a \le x \le 2\,a$, (i.e. when all the antenna is completely on one side of the axis of rotation) the moment of inertia is

$$I_{a_2} = \int_{x-a}^{x+a} dm\, r^2 = \frac{m}{2a}\left[\frac{(x+a)^3}{3} - \frac{(x-a)^3}{3}\right]. \tag{S.33}$$

So these two cases lead to the same expressioon, valid for the general case (i.e. $0 \le x \le 2\,a$), which be then written as

$$I_a\,(x) = \frac{m}{2\,a}\left[\frac{(a+x)^3}{3} + \frac{(a-x)^3}{3}\right] = \frac{m}{2\,a}\,2\,\frac{a^3 + 3\,x^2\,a}{3} = m\frac{a^2 + 3\,x^2}{3}. \tag{S.34}$$

When $x = 0$ we have

$$I_a(x=0) = m\,\frac{a^2}{3}. \tag{S.35}$$

The moment of inertia of the whole system is

$$I(x), = I_M + 2\,[I_a(x) - I_a(0)] = \frac{M\,a^2}{2} + 2\,m\,x^2 \tag{S.36}$$

so that for $x = 0$ it is the moment of inertia of the cylinder $I(0) = I_M$ (the subtraction term $I_a(0)$ are there to take into account that, when $x = 0$, the moment of inertia of the antennae is included in the one of the satellite. The factor two takes is there because of the two antennae).
The potential energy of the spring is

$$U = 2k\,x^2 \tag{S.37}$$

The lagrangian therefore is

$$\mathcal{L} = T - U = 2\,m\,\frac{\dot{x}^2}{2} + \frac{1}{2}I(x)\,\omega^2(x) - 2k\,x^2. \tag{S.38}$$

(b) The total angular momentum is given by

$$J = I\,\omega = \text{const.} \tag{S.39}$$

The fact tha the angular momentum is conserved implies

$$I(2\,a)\,\omega(2a) = I_M\,\bar{\omega} \Rightarrow \omega(x = 2a) = \frac{M\,a^2}{2}\bar{\omega}\left[\frac{M\,a^2}{2} + 8\,m\,a^2\right]^{-1} = \frac{M}{M + 16\,m}\bar{\omega} \tag{S.40}$$

(c) The work done by the spring to completely extend is

$$L = -\int_0^{4\,a} F\,dx = K\int_0^{4\,a} dx\,x = 8\,K\,a^2. \tag{S.41}$$

If no net work is done by the system, energy is conserved , so it is the same for $x = 0$ and $x = 2\,a$, and the antennae are motionless, so that the kinetic energy vanishes. So energy conservation reads:

$$\frac{1}{2}I_M\,\bar{\omega}^2 = \frac{1}{2}I(2a)\;[\omega(2a)]^2 + 8\,K\,a^2 \tag{S.42}$$

so

$$\begin{aligned} K &= \frac{I_M\,\bar{\omega^2} - I(2\,a)\;[\omega(2a)]^2}{16\,a^2} = \frac{I_M\,\bar{\omega}^2}{16\,a^2}\left(1 - \frac{I_M}{I(2a)}\right) \\ &= \frac{I_M\,\bar{\omega}^2}{16\,a^2}\,\frac{8\,m\,a^2}{I_M + 8\,m\,a^2} = \frac{1}{2}\frac{I_M}{I(2a)}m\,\bar{\omega}^2 = \frac{1}{2}\frac{M\,m}{M + 16\,m}\,\bar{\omega}^2 \end{aligned} \tag{S.43}$$

where we have used that

$$I(2a) - I_M = 2\,m\,(2a)^2 = 8\,m\,a^2. \tag{S.44}$$

Exercise 3. ***The Isoperimetric Problem***

In this exercise we want to find among all closed curves with a given perimeter L, which one encloses the greatest area A?

(a) Parametrize the curves by $t \mapsto (x(t), y(t))$ with $t \in [0, 1]$ and show that

$$L = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2} dt \qquad \text{and} \qquad A = \frac{1}{2} \int_0^1 (x\dot{y} - y\dot{x}) dt, \tag{3}$$

where $\dot{x} = \frac{dx}{dt}$.

(b) Using the calculus of variations determine which curve maximizes the area A for a given perimeter L.

Hint. Note that we have a variational problem with an integral constraint. In order to see how to map this problem into an unconstraint one, check Exercise 3 (The Suspension Bridge) of Exercise Sheet 4.

Solution.

(a) Given the parametrization of the curve as $t \mapsto (x(t), y(t))$, the line element is $dl = \sqrt{dx^2 + dy^2} = \sqrt{\dot{x}^2 + \dot{y}^2} dt$. Hence the first equation for L.

The area formula follows directly from the Green's theorem (which is the curl theorem in the plane). According to the Green's theorem, over a region M in the plane with boundaries ∂M, we have

$$\oint_{\partial M} P(x, y)dx + Q(x, y)dy = \iint_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \tag{S.45}$$

where P and Q are functions of (x, y) defined on an open region containing M and have continuous partial derivatives on that region.

Now if we choose $P(x, y) = -y/2$ and $Q(x, y) = x/2$, we have $\partial Q/\partial x - \partial P/\partial y = 1$ and

$$A = \iint_M dxdy = \iint_M \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_{\partial M} P(x, y)dx + Q(x, y)dy = \frac{1}{2} \oint_{\partial M} -ydx + xdy \tag{S.46}$$

$$= \frac{1}{2} \int_0^1 (x\dot{y} - y\dot{x}) dt. \tag{S.47}$$

In the last step we used the parametrization of the curve.

(b) From Exercise 3 (The Suspension Bridge) of Exercise Sheet 4, we know that we should form the function

$$F^\lambda(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(x\dot{y} - y\dot{x}) - \lambda(\sqrt{\dot{x}^2 + \dot{y}^2} - L) \tag{S.48}$$

and look for extrema of the unconstraint functional

$$\int_0^1 F^\lambda(x, y, \dot{x}, \dot{y}) dt. \tag{S.49}$$

We can find the extrema by solving the Euler-Lagrange equations:

$$\frac{\partial F^\lambda}{\partial x} - \frac{d}{dt} \left(\frac{\partial F^\lambda}{\partial \dot{x}} \right) = 0, \qquad \frac{\partial F^\lambda}{\partial y} - \frac{d}{dt} \left(\frac{\partial F^\lambda}{\partial \dot{y}} \right) = 0. \tag{S.50}$$

Theses are

$$\frac{1}{2}\dot{y} - \frac{d}{dt} \left(-\frac{1}{2}y - \lambda \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0, \tag{S.51}$$

$$-\frac{1}{2}\dot{x} - \frac{d}{dt} \left(\frac{1}{2}x - \lambda \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 0. \tag{S.52}$$

Integrating and multiplying the first one by $\dot{y}$ and the second one by $\dot{x}$ gives:

$$\frac{1}{2}y\dot{y} - \left(-\frac{1}{2}y - \lambda\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)\dot{y} + C_1\dot{y} = 0, \tag{S.53}$$

$$-\frac{1}{2}x\dot{x} - \left(\frac{1}{2}x - \lambda\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)\dot{x} + C_2\dot{x} = 0. \tag{S.54}$$

Above C_1 and C_2 are integration constants. Subtracting the last equations gives:

$$y\dot{y} + x\dot{x} + C_1\dot{y} - C_2\dot{x} = \frac{1}{2}\frac{d}{dt}\left[(x-C_2)^2 + (y+C_1)^2\right] = 0, \tag{S.55}$$

which is equivalent to the equation of a circle.

$$(x-C_2)^2 + (y+C_1)^2 = \text{const} = R^2. \tag{S.56}$$

Exercise 4. ***Rotating Asymmetric Top***

Show that the total torque on an object in homogeneous gravitational field vanishes if we choose the reference point in its centre of mass.

Consider an asymmetric object with principal moments of inertia $I_1 < I_2 < I_3$. It originally rotates completely about the first axis such that the angular velocities are $\omega_1 = \Omega$, $\omega_2 = \omega_3 = 0$. A small perturbation causes the angular velocities to change as

$$\omega_1 = \Omega + \eta_1(t),\ \omega_2 = \eta_2(t),\ \omega_3 = \eta_3(t).$$

Show from the Euler's equation by neglecting terms $\mathcal{O}(\eta^2)$ that the subsequent motion is described by

$$\ddot{\eta}_2 = A\eta_2$$

for some constant $A < 0$ that you should determine and find a similar equation for η_3. Conclude that the rotation about the first axis is stable. What happens if the object originally rotates about a different axis? Can you identify an axis for which the rotation is unstable?

You can now verify what you have found by taking your favorite book on classical mechanics and throwing it in the air.

Solution. The total torque on a body with respect to its centre of mass is

$$\vec{K} = \int_V d^3\vec{r}\,\rho(\vec{r})\,\vec{r}\times\vec{g} = \int_V d^3\vec{r}\,\rho(\vec{r})\,\vec{r}\times\vec{g} = \left(\int_V d^3\vec{r}\,\rho(\vec{r})\,\vec{r}\right)\times\vec{g}.$$

But the definition of the centre of mass tells us that the expression in the brackets is zero!

The total torque on the body is $\vec{K} = 0$ and the Euler's equations become

$$I_1\dot{\omega}_1 = -\omega_2\omega_3(I_3 - I_2), \tag{S.57}$$

$$I_2\dot{\omega}_2 = -\omega_3\omega_1(I_1 - I_3), \tag{S.58}$$

$$I_3\dot{\omega}_3 = -\omega_1\omega_2(I_2 - I_1). \tag{S.59}$$

Substituting $\omega_1 = \Omega + \eta_1(t),\ \omega_2 = \eta_2(t),\ \omega_3 = \eta_3(t)$ and neglecting terms quadratic in η gives $\dot{\eta}_1 = 0$ and

$$I_2\dot{\eta}_2 = -\eta_3\Omega(I_1 - I_3), \tag{S.60}$$

$$I_3\dot{\eta}_3 = -\eta_2\Omega(I_2 - I_1). \tag{S.61}$$

Differentiating the first equation and substituting $\dot{\eta}_3$ from the second one gives

$$\ddot{\eta}_2 = -\dot{\eta}_3\Omega\frac{I_1 - I_3}{I_2} = \eta_2\Omega^2\frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3}. \tag{S.62}$$

Since $I_2 > I_1$ and $I_3 > I_1$ we see that

$$A = \Omega^2 \frac{(I_1 - I_3)(I_2 - I_1)}{I_2 I_3} < 0. \tag{S.63}$$

Assuming $\dot{\eta}_2(0) = 0$ the solution is $\eta_2(t) = \eta(0) \cos \sqrt{|A|} t$, which means that the amplitude of the perturbation remains small and so the rotation about the first axis is stable.

On the other hand if the object initially rotates about the second axis such that the initial conditions are $\omega_1 = \eta_1(t),\ \omega_2 = \Omega + \eta_2(t),\ \omega_3 = \eta_3(t)$, similar steps lead to $\ddot{\eta}_1 = B\eta_1$ where

$$B = \Omega^2 \frac{(I_2 - I_1)(I_3 - I_2)}{I_1 I_3} > 0. \tag{S.64}$$

This time the solution becomes $\eta_1(t) = \eta_1(0) \cosh \sqrt{B} t$ and we see that the perturbations do not remain small. This means that the rotation about the principal axis with the intermediate moment of inertia is unstable.