Exercise 1. ***From Coupled Oscillators to Wave Equation***

Consider a ring of N masses m free to oscillate about their respective equilibrium positions, coupled by springs of spring constant κ. Let ϕ_k denote the displacement of the k-th mass from its equilibrium position. We impose periodic boundary conditions, i.e.: $\phi_{k+N} = \phi_k$ and *neglect the curvature of the chain*, such that it behaves as a *1D chain of masses* where the last mass couples to the first.

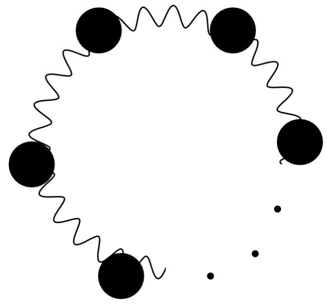

Figure 1: A ring of N masses

i) Show that the Lagrangian of the system is given by:

$$\mathcal{L} = \sum_{k=1}^{N} \left(\frac{1}{2} m \dot{\phi}_k^2 - \frac{1}{2} \kappa (\phi_k - \phi_{k+1})^2 \right)$$

Be especially careful not to overcount the potential terms twice, i.e. note that:

$$\sum_{k=1}^{N} \frac{1}{2} \kappa ((\phi_k - \phi_{k+1})^2 + (\phi_k - \phi_{k-1})^2) = \sum_{k=1}^{N} \kappa (\phi_k - \phi_{k+1})^2$$

since you can shift the indices under the sum thanks to the boundary conditions (convince yourself about it).

Solution. Observe that the potential induced by spring depends only on relative displacements of nearest neighbours from their equilibrium position. The potential induced by the k-th spring is then given by:

$$V = \frac{1}{2} \kappa (\phi_k - \phi_{k+1})^2$$

and therefore

$$\mathcal{L} = \sum_{k=1}^{N} \left(\frac{1}{2} m \dot{\phi}_k^2 - \frac{1}{2} \kappa (\phi_k - \phi_{k+1})^2 \right)$$

ii) Convince yourself that the equation of motion for the k-th mass is given by:

$$\ddot{\phi}_k = \frac{\kappa}{m} (\phi_{k-1} - 2\phi_k + \phi_{k+1})$$

If in doubt, use force!

Solution. Use Euler Lagrange equations and be (very) careful with the sum - i.e. note that:

$$\frac{\partial L}{\partial \phi_q} = \kappa(\phi_{q+1} - \phi_q) - \kappa(\phi_q - \phi_{q-1}) = \kappa(\phi_{q-1} - 2\phi_q + \phi_{q+1})$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_q}\right) = m\ddot{\phi}_k \tag{S.1}$$

The generalised coordinates correspond to position and linear momenta in this situation, so the generalised force is just a force and you can get this equation fast by considering Newton's 2nd law.

iii) Make an ansatz for the solution of the form $\phi_k \propto \exp(i(\varsigma k + \omega t))$, where ς, ω are constants that can be interpreted as wavenumber and (angular) frequency respectively (why?). Find the expression for ω by inserting your ansatz into the equations. You should find:

$$\omega = \pm 2\sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{\varsigma}{2}\right)\right|$$

Note 1: This tells you that for every choice of ς, there are two ω frequency solutions (with the exception of $\varsigma = 0$ for which the solutions are degenerate), one traveling to the "left" and the other to the "right". You should give an argument for what the "traveling" means.

Solution. Inserting the ansatz into the equation of motion, one obtains:

$$\ddot{\phi_k} = -\omega^2\phi_k = \frac{\kappa}{m}(\exp(-i\varsigma) + \exp(i\varsigma) - 2)\phi_k = \frac{\kappa}{m}(2\cos(\varsigma) - 2)\phi_k = -\frac{\kappa}{m}4\sin^2\left(\frac{\varsigma}{2}\right)\phi_k$$

$$\omega^2 = \frac{\kappa}{m}4\sin^2\left(\frac{\varsigma}{2}\right)$$

iv) By recalling that we are dealing with a string of masses and therefore $\phi_{k+N} = \phi_k$, convince yourself that there is a countably finite set of allowed ς.

Note 2: These correspond to different normal frequencies of your chain - and each ansatz function with such allowed ς corresponds to a normal mode on your chain.

Solution. We expect $\phi_{k+N} = \phi_k$ and therefore

$$\exp(i(\varsigma(k+N) + \omega t)) = \exp(i(\varsigma k + \omega t))$$

Hence $\exp(i\varsigma N) = 1$ and $\varsigma = 2\pi k$ for $k \in \mathbb{Z}$. There are countably many integers and therefore there are countably many ςs.

v) Consider $N \to \infty$, while keeping the overall circumference of the mass ring constant. In such case $a \to 0$ and one can approximate the index k by a continuous position "label" $x = ka$: $\phi_k(t) \to \phi(ka, t) = \phi(x, t)$. Convince yourself that the equation of motion now reads:

$$\frac{\partial^2\phi(x,t)}{\partial t^2} = \frac{\kappa}{m}(\phi(x-a,t) - 2\phi(x,t) + \phi(x+a,t))$$

iv) Taylor expand $\phi(x \pm a)$ around x to the second order to show that:

$$\frac{\partial^2\phi(x,t)}{\partial t^2} = \frac{\kappa a^2}{m}\frac{\partial^2\phi(x,t)}{\partial x^2}$$

Consider a case where $\kappa \to \infty$ as we go to $a \to 0$, in a way that $\frac{\kappa a^2}{m} = const. = c^2$. Have you seen such equation before? What is the interpretation of the constant c? Can you justify your motivation by some clever argument? If you are keen enough, check that $f(x \pm ct)$ is a solution to this equation for any sufficiently smooth function f.

Solution.

$$\phi(x-a,t) - 2\phi(x,t) + \phi(x+a,t) = 2\phi(x,t) - \phi'(x,t)a + \phi'(x,t)a - 2\phi(x,t) + \phi''(x,t)a^2 = \phi''(x,t)a^2$$

where $\phi' = \frac{\partial \phi}{\partial x}$. It is a wave equation, c is the phase velocity of the wave propagation. There was an error in the original question - you might have found that the constant should have been defined as $\frac{\kappa a^2}{m} = c^2$ and not $\frac{\kappa a^2}{m} = c^{-2}$. We will stick to $\frac{\kappa a^2}{m} = c^2$. Apologies for that. As an argument to interpret what c is to consider any function $f(x \pm ct)$. This solves the equation:

$$\frac{\partial^2 \phi(x,t)}{\partial t^2} = c^2 \frac{\partial^2 \phi(x,t)}{\partial x^2}$$

Consider how a point on this function "moves" in x as you increase the time - you should find that in unit time, it moves by $\pm c$. So the constant is related to the "propagation velocity" of your solution (a waveform). Take the remaining part as a bonus and do not get too worried if you can not follow it as a whole - it is here only to link the f solution to the notion of the Fourier transform. Since your initial ansâtze of $\exp(i(\varsigma k + \omega t))$ form a complete orthogonal set of functions, you can Fourier compose your $f(x \pm ct)$ through:

$$\varsigma' = \frac{\varsigma}{a}$$
$$f(x \pm ct) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\varsigma') \exp(-i(\varsigma' x \pm c\varsigma' t)) d\varsigma'$$

where $\tilde{f}(\varsigma')$ is a fourier transform of the traveling waveform and $\exp(-i(\varsigma' x \pm c\varsigma' t))$ is our single mode ansantz for $\omega = \pm c\varsigma'$ - i.e. a single mode obeying the above differential equation. This relates your original ansatz to the f solution and gives an argument why any smooth $f(x \pm ct)$ should be a solution, given we know that $\exp(-i(\varsigma' x \pm c\varsigma' t))$ is. Particularly note that by fixing the speed of your waveform to c, you got rid of the fourier transform over ω.

Note 3: If you cannot recall the equation, remind yourself what a wave equation looks like, as you will use it (with some damping/dissipative term) in one of the following exercises.

Note 4: The coupled harmonic oscillator chain is a standard way to give a motivation for a notion of field. As you will see in your advanced courses such as General Relativity or Quantum Field Theory, the field viewpoint provides enormous insight into a lot physical phenomena.

Exercise 2. *Moments of inertia*

The Huygens-Steiner theorem (sometimes also known as "parallel axis theorem") is an easy-to-prove result that relates moments of inertia for rotations of the same rigid body around any axis to the one around a parallel axis through its center of mass.

a) Let $\mathcal{I}_a$ be the moment of inertia of a rigid body with mass m for rotations around an axis a that goes through its center of mass. Show that the moment of inertia $\mathcal{I}_b$ for rotations around any axis b parallel to a can be obtained from $\mathcal{I}_a$ through the formula

$$\mathcal{I}_b = \mathcal{I}_a + MR^2, \tag{1}$$

where R is the distance between a and b.

Solution. Describe the body with a generic density distribution $\rho(\vec{r})$ with

$$M = \int \rho(\vec{r}) d^3\vec{r}, \tag{S.2}$$

and choose the reference frame so that the body's center of mass coincides with the origin

$$\vec{r}_{\text{CM}} \equiv \frac{1}{M} \int \rho(\vec{r})\, \vec{r}\, d^3\vec{r} = \vec{0}. \tag{S.3}$$

Furthermore, choose the frame's orientation so that the z axis coincides with a and b intersects the x axis at $x = -R$. One then finds

$$\mathcal{I}_a = \int \rho(\vec{r})(x^2 + y^2) d^3\vec{r}, \tag{S.4}$$

for the first moment of inertia, and

$$\begin{aligned}\mathcal{I}_b &= \int \rho(\vec{r})[(x+R)^2 + y^2] d^3\vec{r} = \\ &= \int \rho(\vec{r})[x^2 + 2xR + R^2 + y^2] d^3\vec{r} = \\ &= \int \rho(\vec{r})[x^2 + y^2] d^3\vec{r} + R^2 \int \rho(\vec{r}) d^3\vec{r} + R \int \rho(\vec{r})\, x\, d^3\vec{r}.\end{aligned} \tag{S.5}$$

Because of (S.3) the last integral vanishes, and it is immediate to recognise the first two correspond to the terms in equation (1).

b) Compute the moment of inertia for a uniform ball of mass M and radius R with respect to an axis that goes through its center. Then use the Huygens-Steiner theorem to see how the result changes when the ball rotates around an axis at $R/2$ from its center.

Solution. Let

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}, \tag{S.6}$$

be the density of the ball; working in spherical coordinates[1] its moment of inertia is therefore given by

$$\begin{aligned}\mathcal{I}^{\text{ball}}_{\text{CM}} &= \int_{r<R} \rho\, (r\sin\theta)^2\, d^3\vec{x} = \rho \int_{r<R} r^4 \sin^2\theta\, dr\, d\cos\theta\, d\phi = \\ &= 2\pi\rho \int_0^R r^4\, dr \int_{-1}^{+1} (1-\cos^2\theta) d\cos\theta = 2\pi\rho \frac{R^5}{5}\frac{4}{3} = \frac{2}{5} M R^2.\end{aligned} \tag{S.7}$$

At $R/2$ the parallel axis theorem gives

$$\mathcal{I}^{\text{ball}}_{R/2} = \mathcal{I}^{\text{ball}}_{\text{CM}} + M\left(\frac{R}{2}\right)^2 = \frac{13}{20} M R^2. \tag{S.8}$$

c) Compute the moment of inertia for a thin spherical shell of mass M and radius R with respect to an axis passing through its center. Compare with the previous result.

Solution. Let

$$\sigma = \frac{M}{4\pi R^2}, \tag{S.9}$$

be the surface density of the sphere; working in spherical coordinates on its surface its moment of inertia is therefore given by

$$\mathcal{I}^{\text{sphere}}_{\text{CM}} = \int \sigma\, (R\sin\theta)^2\, d^2\Sigma = \sigma R^4 \int \sin^2\theta\, d\cos\theta\, d\phi = 2\pi\sigma R^4 \frac{4}{3} = \frac{2}{3} M R^2. \tag{S.10}$$

Clearly this is greater than the ball *for the same total mass*, because matter is concentrated further from the rotation axis.

d) Compute the moment of inertia for a spherical cloud of mass M and density proportional to e^{-r}, always with respect to an axis through its center.

[1] Clearly any other coordinate choice (e.g. cylindrical) leads to the same result.

Only this solution is provided here, but it should not be hard to find references for these calculations.

Solution. Let

$$\rho(r) = c\, e^{-r/R}, \tag{S.11}$$

be the cloud's density, and work once more in spherical coordinates. The constant can be fixed by requiring

$$M = \int_0^\infty \vec{x}^2 \rho(|\vec{x}|) d^3\vec{x} = 4\pi \int_0^\infty r^2 \rho(r) dr = 4\pi c \int_0^\infty r^2 e^{-r/R} dr. \tag{S.12}$$

In order to compute the integral one may observe that, thanks to the exponential, integration by parts is particularly effective:

$$\int_0^\infty r^n e^{-r/R} dr = \left[-R r^n e^{-r/R} \right]_0^\infty + nR \int_0^\infty r^{n-1} e^{-r/R} dr = \begin{cases} R & \text{if} \quad n = 0; \\ nR \int_0^\infty r^{n-1} e^{-r/R} dr & \text{otherwise.} \end{cases},$$

$$\int_0^\infty r^n e^{-r/R} dr = n! R^{n+1} \,. \tag{S.13}$$

The constant is thus fixed by $c = M/8\pi R^3$. The moment of inertia may then be found as

$$\mathcal{I}_{\text{CM}}^{\text{cloud}} = 4\pi c \int r^4 e^{-r} \sin^2\theta \, dr \, d\cos\theta = (4\pi c)(4! R^5) \frac{2}{3} = 8MR^2. \tag{S.14}$$

Exercise 3. ***Oscillating string with friction.***

A uniform string has length L and mass per unit length ρ. It undergoes small transverse vibration in the (x, y) plane with its endpoints held fixed at $(0, 0)$ and $(L, 0)$ respectively. The tension is K. The string is subject to a small velocity-dependent frictional force $-kv\delta l$ to each small piece of length δl with transverse velocity v. Using appropriate approximations, the following equations hold for the vibration amplitude $y(x, t)$:

$$\frac{\partial^2 y}{\partial t^2} + a\frac{\partial y}{\partial t} = b\frac{\partial^2 y}{\partial x^2} \tag{2}$$

$$y(0, t) = 0 = y(L, t) \tag{3}$$

(a) Find the constants a and b in (2).

(b) Find all solutions of (2) and (3) which have the product form $y = X(x)T(t)$. You may assume $a^2 < b/L^2$.
(Hint: The wave equation (2) is separable, i.e. it can be written as $F(X'', X', X, x) = G(T'', T', T, t)$. In order for the equation to hold for all x and t, each side must be equal to a constant, which is taken as $-\lambda^2$. Solve the two equations for $T(t)$ and $X(x)$ separately, using the boundary conditions (3).)

Solution.

(a) The frictional force acting on unit length of the string is $-kv = -k\partial y/\partial t$, so the transverse vibration of the string is described by

$$\rho\frac{\partial^2 y}{\partial t^2} = K\frac{\partial^2 y}{\partial x^2} - k\frac{\partial y}{\partial t} \tag{S.15}$$

or

$$\frac{\partial^2 y}{\partial t^2} + \left(\frac{k}{\rho}\right)\frac{\partial y}{\partial t} = \left(\frac{K}{\rho}\right)\frac{\partial^2 y}{\partial x^2}\ . \tag{S.16}$$

Hence, $a = k/\rho$, $b = K/\rho$.

Note: This can be easily derived from one-dimensional wave equation with a dampening factor $F_d = -k\frac{\partial y}{\partial t}L$:

$$M\frac{\partial^2 y}{\partial t^2} - \kappa L^2\frac{\partial^2 y}{\partial x^2} = F_d\ , \tag{S.17}$$

where L is the length of the string, $M = L\rho$ is its mass and $\kappa = K/L$ its stiffness.

(b) Setting $y = X(x)T(t)$ and substituting it in the wave equation (2) we obtain

$$T''X + aT'X = bX''T \Rightarrow \frac{T''}{T} + \frac{aT'}{T} = \frac{bX''}{X}\ . \tag{S.18}$$

As the left-hand side depends only on t and the right-hand side depends only on x, each must be equal to a constant, say $-b\lambda^2$. Thus we have

$$\frac{bX''}{X} = -b\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \tag{S.19}$$

$$\frac{T''}{T} + \frac{aT'}{T} = -b\lambda^2 \Rightarrow T'' + aT' + b\lambda^2 T = 0\ . \tag{S.20}$$

Using the boundary conditions (3)

$$y(0, t) = y(L, t) = 0 \Rightarrow X(0) = X(L) = 0 \tag{S.21}$$

we obtain the solutions for the first equation (S.19)

$$X_n(x) = A_n\sin(\lambda_n x) = A_n\sin(\frac{n\pi x}{L}) \tag{S.22}$$

where $\lambda_n = \frac{n\pi}{L}$, A_n is a constant and $n = 1, 2, 3...$. The second equation (S.20) then becomes

$$T'' + aT' + b\left(\frac{n\pi}{L}\right)^2 T = 0 \, .$$

Letting $T = e^{pt}$ we obtain the characteristic equation

$$p^2 + ap + \frac{n^2\pi^2 b}{L^2} = 0$$

whose solutions are

$$p_\pm = \frac{-a \pm \sqrt{a^2 - 4n^2\pi^2 b/L^2}}{2} = -\frac{a}{2} \pm i\omega_n$$

where

$$\omega_n = \sqrt{\frac{n^2\pi^2 b}{L^2} - \frac{a^2}{4}}$$

is real as $\pi^2 b/L^2 > a^2/4$. Hence the solution of the second equation can be written as

$$T_n = [C'_n \sin(\omega_n t) + D'_n \cos(\omega_n t)]e^{-\frac{at}{2}}$$

and thus

$$y_n = \sin\left(\frac{n\pi x}{L}\right)[C_n \sin(\omega_n t) + D_n \cos(\omega_n t)]e^{-\frac{at}{2}} \, ,$$

collecting the constants in each term into one, i.e. $C_n = C'_n A_n, D_n = D'_n A_n$. The general solution of the wave equation is thus

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) \, .$$