Exercise 1. ***Laplace-Runge-Lenz vector***

The *Laplace-Runge-Lenz* vector defined as

$$\vec{A} = \vec{p} \times \vec{L} - m\,k\,\frac{\vec{r}}{r} \tag{1}$$

for a body under the influence of potential

$$V(r) = -\frac{k}{r} \tag{2}$$

is constant both in magnitude and direction.

Use these properties, together with angular momentum conservation, to find the equation that relates the radius and the angle describing the orbit, knowing that the magnitude of the LRL vector is

$$A = m\,k\,e\,, \tag{3}$$

where e is the eccentricity of the orbit.
Hint: consider $\vec{A} \cdot \vec{r}$ and use the invariance properties of the Levi-Civita-Symbol

Solution. Let us describe the orbit in polar coordinates, using the radius r and angle θ, the origin being the centre of the orbit. Since the LRL vector is a constant (also in direction) we can define the angle θ as the angle between the position vector $\vec{r}$ and $\vec{A}$ (you have learned that the $\vec{A}$ always points in the direction of the major axis, so θ is also the angle between the position $\vec{r}$ and this axis).

$$\begin{aligned}\vec{A}\cdot\vec{r} = A\,r\,\cos\theta &= \vec{r}\cdot\left(\vec{p}\times\vec{L}\right) - m\,k\,r \\ &= \vec{L}\cdot(\vec{r}\times\vec{p}) - m\,k\,r \\ &= l^2 - m\,k\,r\end{aligned} \tag{S.1}$$

where we have used the fact that the three-dimentional Levi-Civita symbol is invariant under cyclical permutation of its indices

$$\vec{r}\cdot\left(\vec{p}\times\vec{L}\right) = r_i\,\epsilon_{ijk}\,p_j\,L_k = L_k\,\epsilon_{kij}\,r_i\,p_j = \vec{L}\cdot(\vec{r}\times\vec{p}) \tag{S.2}$$

$$A\,r\,\cos\theta = l^2 - m\,k\,r \quad\Rightarrow\quad \frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right) \tag{S.3}$$

which can be rewritten as

$$\Rightarrow\quad \frac{1}{r} = \frac{mk}{l^2}\,(1 + e\,\cos\theta) \tag{S.4}$$

Exercise 2. ***Perihelion precession***

Discuss the motion of a particle of reduced mass μ in the following central forces field,

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \tag{4}$$

This force is composed of the gravitation force and of a perturbing term proportional to the inverse cubic of the relative distance r. Assume that the additional force is very small compared to the gravitational force.

a) Show that the equation of the orbit can be cast in the following form

$$r = r(\phi) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos(\alpha\phi)} \tag{5}$$

where r, ϕ are the polar coordinates in the relative coordinate frame as discussed in the lecture. The parameters are $a = -k/2E$, where E is the energy of the orbit, while ε is the eccentricity of the orbit to be determined. The parameter α results from the presence of the perturbing term in the force and is responsible for a motion of *precession* (see next part). Find the values of ε and α in terms of the other constants of the problem.

Hint. Recall the form of the effective potential, which is valid for every central force.

The orbit (7) is closed and it is an ellipse when $\alpha = 1$, which correspondes to the Kepler problem (i.e. the case of $C = 0$). When $\alpha \neq 1$ the orbit is not closed and it is an ellipse with a motion of *precession*. The precession can be described by rotation of the *perihelion*, which is the inversion point of the orbit.

b) Write down the effective one-dimensional equation of motion for the central field (4) and convince yourself that it has the same form as the one for the Kepler problem, but with an augmented angular momentum. Relate the augmented part of the angular momentum into the precession of the elliptical orbit and show that the *angular speed* ω_p *of the precession* is

$$\omega_p = \frac{\pi}{T}\frac{1}{1-\varepsilon^2}\frac{C}{ka}, \tag{6}$$

where T is the period of the orbit.

Note that ω_p is expressed in terms of dimensionless parameter $\eta = C/ka$, which is the measure of the perturbation relative to the gravity.

c) The experimental observations have shown that the perihelion of Mercury precesses at a rate of $\omega_p/2\pi = 40$ arcseconds per century. Show that such a precession would require $\eta \approx 1.42 \cdot 10^{-7}$.

Hint. The eccentricity of the orbit of Mercury is $\varepsilon = 0.206$ *and the period is* $T = 0.24$ *years.*

Solution.

a) We first obtain an expression of the potential energy associated to the force (4). We write the line integral making use of the fact that the gravitational force is directed inwards (radially) and the inverse cubic term outwards (radially), i.e.,

$$U = -\int_\infty^r \langle \vec{F}, \vec{ds}\rangle = \int_\infty^r \frac{k}{r^2}dr - \int_\infty^r \frac{C}{r^3}dr = -\frac{k}{r} + \frac{C}{2r^2} \tag{S.5}$$

where we set $U(\infty) = 0$. Making use of the conservation of angular momentum, as we saw in the lecture and in the previous exercises, the total energy of the particle can be written as

$$E = T + U_{eff} = \frac{\mu}{2}\dot{r}^2 - \frac{k}{r} + \frac{C}{2r^2} + \frac{L^2}{2\mu r^2} \tag{S.6}$$

where L is the modulus of the angular momentum and μ the reduced mass. We see that the perturbing force renormalizes the centrifugal angular momentum contribution to the effective potential, and we can write

$$U_{eff}(r) = -\frac{k}{r} + \frac{C}{2r^2} + \frac{L^2}{2\mu r^2} = -\frac{k}{r} + \frac{L^2}{2\mu r^2}(1+\delta) \tag{S.7}$$

where $\delta = \frac{C\mu}{L^2}$. Setting $\delta = 0$ we recover the pure gravitational case. As we have seen in the lecture, we can obtain the equation of the orbit by solving the integral

$$\phi(r) - \phi(r_0) = \pm \int_{r_0}^{r} dx \frac{L}{x^2} \frac{1}{\sqrt{2\mu[E - U_{eff}(x)]}} = \pm \int_{r_0}^{r} dx \frac{L}{x^2} \frac{1}{\sqrt{2\mu E + 2\mu k/x - L^2(1+\delta)/x^2}} \tag{S.8}$$

where E is the energy of the orbit. With the substitution $x = 1/s$ the integral above becomes

$$\phi(r) - \phi(r_0) = \mp \int_{1/r_0}^{1/r} ds \frac{1}{\sqrt{2\mu E/L + 2\mu k s/L^2 - (1+\delta)s^2}} \tag{S.9}$$

We now need to play with the integrand to bring the integral in the form $\int dy(1-y^2)^{-1/2} = -\arccos(y)$. By completing the square under the square root, we write the denominator of the integrand as

$$\sqrt{\frac{2\mu E}{L} + \frac{2\mu k s}{L^2} - (1+\delta)s^2} = \sqrt{\frac{2\mu E L^2(1+\delta) + \mu^2 k^2}{L^4(1+\delta)}} \sqrt{1 - \frac{L^4(1+\delta)}{2\mu E L^2(1+\delta) + \mu^2 k^2} \left(s\sqrt{1+\delta} - \frac{\mu k}{L^2\sqrt{1+\delta}}\right)^2} \tag{S.10}$$

By making first the variable substitution

$$z = s\sqrt{1+\delta} - \frac{\mu k}{L^2\sqrt{1+\delta}} \tag{S.11}$$

and then

$$y = \frac{L^2\sqrt{1+\delta}}{\sqrt{2\mu E L^2(1+\delta) + \mu^2 k^2}} z \tag{S.12}$$

we obtain

$$\phi(r) - \phi(r_0) = \mp \frac{1}{\sqrt{1+\delta}} \int_{y_0}^{y_r} dy \frac{1}{\sqrt{1-y^2}} \tag{S.13}$$

with

$$y_r = \frac{L^2(1+\delta)/r - \mu k}{\sqrt{2\mu E L^2(1+\delta) + \mu^2 k^2}}. \tag{S.14}$$

The final integration thus yields

$$\phi(r) = \frac{1}{\sqrt{1+\delta}} \arccos\left[\frac{L^2(1+\delta)/r - \mu k}{\sqrt{2\mu E L^2(1+\delta) + \mu^2 k^2}}\right] \tag{S.15}$$

The argument in the inverse cosine can be written as

$$\frac{L^2(1+\delta)/r - \mu k}{\sqrt{2\mu E L^2(1+\delta) + \mu^2 k^2}} = \frac{\frac{L^2}{\mu k r}(1+\delta) - 1}{\sqrt{1 + \frac{2EL^2}{\mu k^2}(1+\delta)}} = \frac{1}{\varepsilon}\left[\frac{L^2}{\mu k r}(1+\delta) - 1\right] = \frac{1}{\varepsilon}\left[-\frac{k}{2Er}(1-\varepsilon^2) - 1\right] \tag{S.16}$$

where we introduced the eccentricity of the orbit $\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}(1+\delta)}$. Note that ε equals the Kepler case when $\delta = 0$. Placing this result back in the orbit equation we obtain

$$\phi(r) = \frac{1}{\sqrt{1+\delta}} \arccos\left[\frac{-\frac{k}{2Er}(1-\varepsilon^2) - 1}{\varepsilon}\right] \tag{S.17}$$

which yields

$$\varepsilon \cos(\alpha\phi) = -\frac{k}{2Er}(1-\varepsilon^2) - 1 \tag{S.18}$$

and finally

$$r = \frac{a(1-\varepsilon^2)}{1 + \varepsilon\cos(\alpha\phi)} \tag{S.19}$$

where $a = -k/2E$ and $\alpha = \sqrt{1+\delta}$.

a) ***Alternative solution*** Starting from the result of problem 3, set 7 the equation for $u(\theta) = 1/r(\theta)$ is

$$u'' + u = -\frac{\mu}{L^2}\frac{f(1/u)}{u^2}.$$

In our case one can write $f(1/u) = -\mu k u^2 + \mu C u^3$ (we redefined the constants $k \to \mu k$, $C \to \mu C$ for convenience.) We define further the angular momentum per unit mass $\ell = L/\mu$. This leads to the harmonic oscillator equation for u

$$u'' + u = \frac{k}{\ell^2} - \frac{C}{\ell^2}u,$$

or alternatively

$$u'' + \alpha^2 u = \frac{k}{\ell^2},$$

where $\alpha^2 = 1 + C/\ell^2$. The solution is simply

$$u = \frac{k}{\ell^2\alpha^2}\left[1 + \varepsilon\cos\alpha(\theta - \theta_0)\right]$$

where ε is for the moment an unknown constant. One can set $\theta_0 = 0$ by demanding that (for $\varepsilon > 0$) there is a maximum of $u(\theta)$ and hence minimum of $r(\theta)$ at $\theta = 0$. Then

$$r(\theta) = \frac{\ell^2\alpha^2}{k(1 + \varepsilon\cos\alpha\theta)}. \tag{S.20}$$

In order to get the solution in the desired form, one has to find a relationship between the angular momentum ℓ, the total energy per unit mass $e = E/\mu$ and the eccentricity ε. The total energy is conserved, so we may find its value at $\theta = 0$ where $r' = \dot{r} = 0$ (remember, $r(\theta)$ has a local minimum at this point!)

$$e = \frac{1}{2}r^2\dot{\theta}^2 - \frac{k}{r} + \frac{C}{2r^2} = \frac{1}{2r^2}\left(\ell^2 + C\right) - \frac{k}{r} = \frac{\ell^2\alpha^2}{2r^2} - \frac{k}{r}.$$

In the second step we used the angular momentum conservation $r^2\dot{\theta} = \ell$. Substituting $r(0) = \ell^2\alpha^2/[k(1+\varepsilon)]$ gives

$$e = \frac{k^2}{2\ell^2\alpha^2}(1+\varepsilon)^2 - \frac{k^2}{\ell^2\alpha^2}(1+\varepsilon) = \frac{k^2}{2\ell^2\alpha^2}(\varepsilon^2 - 1).$$

Taking ℓ^2 from this result and substituting to (S.20) finally gives

$$r(\theta) = \frac{-k/2e(1-\varepsilon^2)}{1 + \varepsilon\cos\alpha\theta}.$$

b) Using the effective potential given in (S.7), we can write the effective one-dimensional equation of motion in the following form

$$\mu\ddot{r} = -\frac{\partial}{\partial r}U_{\text{eff.}}(r) = -\frac{k}{r^2} + \frac{C}{r^3} + \frac{L^2}{\mu r^3} = -\frac{k}{r^2} + \frac{L^2 + \mu C}{\mu r^3} \tag{S.21}$$

$$= -\frac{k}{r^2} + \frac{L^2 + \mu C + (\mu C/2L)^2 - (\mu C/2L)^2}{\mu r^3} = -\frac{k}{r^2} + \frac{(L + \mu C/2L)^2 - (\mu C/2L)^2}{\mu r^3} \tag{S.22}$$

If $\mu C \ll L^2$ (which is the same as $\delta \ll 1$), we can neglect the term $(\mu C/2L)^2$ in comparison with L^2 term, and write

$$\mu\ddot{r} = -\frac{k}{r^2} + \frac{[L + (\mu C/2L)]^2}{\mu r^3} \tag{S.23}$$

Note that this is just the standart equation of motion for the Kepler problem, but with the angular momentum L augmented by $\Delta L = \mu C/2L$. Such an augmentation of the angular momentum can be accounted for by augmenting the angular velocity:

$$L = \mu r^2\dot{\phi} \qquad \longrightarrow \qquad L\left(1 + \frac{\mu C}{2L^2}\right) = \mu r^2\dot{\phi}\left(1 + \frac{\mu C}{2L^2}\right) = \mu r^2\dot{\phi} + \mu r^2\omega_p, \tag{S.24}$$

where

$$\omega_p = \frac{\mu C\dot{\phi}}{2L^2} = \frac{2\pi\mu C}{2L^2 T} = \frac{\pi\mu C}{L^2 T} \tag{S.25}$$

is the precession frequency. Above we used that $\dot{\phi} = 2\pi/T$, where T is the period of the orbit. Now we can express this frequency in terms of η:

$$\omega_p = \frac{\pi}{T}\frac{\mu}{L^2}C = \frac{\pi}{T}\frac{\mu k a}{L^2}\frac{C}{ka} = \frac{\pi}{T}\frac{1}{1-\varepsilon^2}\frac{C}{ka} \tag{S.26}$$

Above we used that $\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}} = \sqrt{1 - \frac{L^2}{\mu k a}}$ (see the part a)).
In order to see that ω_p is the precession frequency, note that the reference frame in which everything is precessing with angular velocity ω_p and there is no perturbation term, the effective one-dimensional equation of motion has exactly the same form as in the stationary case with the perturbation term, (S.23).

c) Putting the numbers ($\varepsilon = 0.206$, $T = 0.24\text{yr}$, $\omega_p/2\pi = 40''$) we find

$$\eta = \frac{C}{ka} = 2(1-\varepsilon^2)T\frac{\omega_p}{2\pi} = 2(1-0.206^2)\cdot(0.24\text{yr})\cdot 40'' \frac{1}{3600''}\frac{1\text{rotation}}{360^\circ}\frac{1\text{century}^{-1}}{100\text{yr}^{-1}}\text{yr}^{-1} \tag{S.27}$$

$$\approx 1.42\cdot 10^{-7}. \tag{S.28}$$

Exercise 3. *Oscillations on a circle*

Consider three identical masses m, which are connected to each other by three identical springs. Denote the spring constant by k and let the rest length be L. In addition, the motion of the masses is restricted to a circle of radius R, as depicted in Fig. 1.

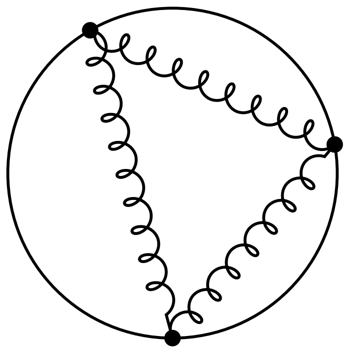

Figure 1: Three masses on sphere, connected by three springs.

a) Choose appropriate generalized coordinates and find the Lagrangian of the system.

Solution. As the motion of the masses is constraint to a circle, polar coordinates offer themselves. We find for the kinetic and potential energies, T and U,

$$T = \frac{1}{2}mR^2\sum_{i=1}^{3}\dot\varphi_i^2\,, \tag{S.29}$$

$$U = \frac{1}{2}k\sum_{i=1}^{3}(\Delta l_i)^2 = \frac{1}{2}k\sum_{i=1}^{3}\left[2R\sin\left(\frac{\varphi_{i+1}-\varphi_i}{2}\right) - L\right]^2, \tag{S.30}$$

where we used the notation $\varphi_4 = \varphi_1$, and with it the Lagrangian

$$L = T - U\,. \tag{S.31}$$

b) Assume that $L = \frac{3}{2}R$ and find a minimum of the energy, in which non of the springs are completely contracted. Linearize the system in a neighborhood of it.

Solution. The energy $T+U$ is minimal at the equilibrium position $\dot\varphi_i = 0$ where the potential energy is minimal. To minimize the potential energy, we introduce the variables $\phi_i = (\varphi_{i+1} - \varphi_i) \in [0, 2\pi)$ and minimize U with respect to them. As the ϕ_i's are not independent we add the constraint

$$\sum_i \phi_i = 2\pi \tag{S.32}$$

by the use of a Lagrange multiplier:

$$0 = \frac{\partial}{\partial\phi_j}(U+\lambda(\phi_1+\phi_2+\phi_3-2\pi)) = kR[2R\sin(\phi_j/2)-L]\cos(\phi_j/2)+\lambda = kR[R\sin(\phi_j)-L\cos(\phi_j/2)]+\lambda\,. \tag{S.33}$$

As we find the same equation for all $j \in \{1,2,3\}$, a possible solution is $\phi_j = \frac{2\pi}{3}$, with $\lambda = \frac{kR}{2}(\sqrt{3}\,R - L)$. To show that this is a minimum, we write $U = U(\phi_1, \phi_2)$ and consider

$$\frac{\partial^2}{\partial\phi_1^2}U\bigg|_{\phi_i=\frac{2\pi}{3}} = 2R^2k\frac{\partial^2}{\partial\phi_1^2}\bigg|_{\phi_i=\frac{2\pi}{3}}\left\{[\sin(\phi_1/2)-3/2]^2+[\sin(\phi_2/2)-3/2]^2+[\sin((2\pi-\phi_1-\phi_2)/2)-3/2]^2\right\} \tag{S.34}$$

$$= kR\frac{\sqrt{3}L-2R}{2} = kR^2\,\frac{3\sqrt{3}-4}{4} > 0\,. \tag{S.35}$$

As we obtain the same for ϕ_2, we see that $\phi_i = 2\pi/3$ is indeed a local minimum. Note that, interestingly, for $L = 0$ we would find a local maximum.

To linearize the system, we expand U in $\Delta\phi_i = 2\pi/3 - \phi_i$,

$$U = \frac{1}{2}k\sum_i[2R\sin(\phi_i/2)-L]^2 \tag{S.36}$$

$$= U_0 + \frac{1}{4}kR^2\sum_i\left\{2[\cos(\phi_i/2)^2-\sin(\phi_i/2)^2]+\frac{L}{R}\sin(\phi_i/2)\right\}\bigg|_{\phi_i=\frac{2\pi}{3}}(\Delta\phi_i)^2+\mathcal{O}(\Delta\phi_i^3) \tag{S.37}$$

$$= U_0 + \frac{1}{2}U_1\sum_i(\Delta\phi_i)^2+\mathcal{O}(\Delta\phi_i^3)\,, \tag{S.38}$$

$$U_1 = \frac{1}{8}kR^2(3\sqrt{3}-4) > 0 \tag{S.39}$$

Neglecting higher order terms and changing back to the initial variables results in

$$U \approx U_0 + \frac{1}{2}U_1\sum_i[2\pi/3-(\varphi_{i+1}-\varphi_i)]^2 = U_0 + \frac{2\pi^2}{3}U_1 + \frac{1}{2}U_1\sum_i(\varphi_{i+1}-\varphi_i)^2\,. \tag{S.40}$$

Therefore, the linearized problem has the form

$$T = \frac{1}{2}mR^2\sum_i\dot\varphi_i^2\,, \tag{S.41}$$

$$U = \frac{1}{2}U_1\sum_i(\varphi_{i+1}-\varphi_i)^2\,, \tag{S.42}$$

where we shifted the potential energy to get rid of the constant terms.

c) Find the eigensolutions of the linearized problem.

Solution. The equation of motion for the linearized problem reads

$$mR^2\ddot\varphi_i = U_1(2\varphi_i-\varphi_{i+1}-\varphi_{i-1})\,, \tag{S.43}$$

or in matrix form

$$\begin{pmatrix}\ddot\varphi_1\\ \ddot\varphi_2\\ \ddot\varphi_3\end{pmatrix} = -\frac{3\sqrt{3}-4}{8}\frac{k}{m}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix}\,. \tag{S.44}$$

The eigenvalues are

$$\lambda_1 = 0\,, \qquad \lambda_2 = \lambda_3 = -3\frac{3\sqrt{3}-4}{8}\frac{k}{m}\,, \tag{S.45}$$

and the corresponding eigenvectors are

$$v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}\,, \qquad v_2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}\,, \qquad v_3 = \begin{pmatrix}1\\-1\\0\end{pmatrix}\,. \tag{S.46}$$

Note that the eigenvectors v_2 and v_3 are not unique, as the eigenvalues are degenerate.

Exercise 4. ***Particle moving on a spiral***

Consider particle of mass m moving under the influence of an attractive central force of magnitude Cm/r^3.

a) Write down the second-order differential equation for r in terms of the effective potential. Show using $u = 1/r$ and the conservation of the angular momentum that it can be written as

$$u'' + \left(1 - \frac{C}{\ell^2}\right) u = 0 \tag{7}$$

where $u' = \mathrm{d}u/\mathrm{d}\theta$ and ℓ is angular momentum per unit mass. *Hint: look at problem 3, sheet 7.*

b) The particle is thrown at point A at distance a from the origin. Initially, it has speed v and moves in direction perpendicular to the line through point A and the origin. Show that if $v^2 < C/a^2$, the particle will spiral towards the origin and give the equation of its trajectory. Show further that it reaches the origin at time

$$T = \frac{a^2}{\sqrt{C - a^2 v^2}}.$$

Hint: you can assume the following identity: $\int_0^\infty \cosh^{-2} x = 1$.

Solution.

a) Starting with the effective equation of motion

$$m\ddot{r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(V(r) + \frac{J^2}{2mr^2}\right) = f(r) + \frac{J^2}{mr^3} \tag{S.47}$$

we use the substitution $u = 1/r$ and replace the time derivative with the derivative wrt. θ. This implies

$$\dot{r} = -\frac{1}{u^2}u'\dot{\theta} = -\frac{J}{m}u',$$

where we used the angular momentum conservation $J = mr^2\dot{\theta}$. Performing the same steps again gives

$$\ddot{r} = -\frac{J^2}{m^2}u^2 u''.$$

Substituting to (S.47) with $f = -Cm/r^3$ implies

$$-\frac{J^2}{m}u^2 u'' = -Cmu^3 + \frac{J^2}{m}u^3.$$

Rewriting this in terms of the angular momentum per unit mass $\ell = J/m$ gives eqn. (7)

b) The angular momentum is $\ell = av$. If $C > \ell^2$, i.e. $v^2 < C/a^2$, then the general solution of (7) can be written as

$$u(\theta) = A\cosh\lambda\theta + B\sinh\lambda\theta,$$

where $\lambda = \sqrt{C/\ell^2 - 1}$. The initial conditions $u(0) = 1/a$ and $u'(0) = 0$ (there is no compoment of the initial velocity pointing towards the origin) imply $A = 1/a$ and $B = 0$. The equation of the orbit then becomes

$$r(\theta) = \frac{1}{u} = \frac{a}{\cosh\lambda\theta}.$$

The total time to reach the origin is

$$T = \int_0^\infty \frac{\mathrm{d}\theta}{\dot{\theta}} = \int_0^\infty \mathrm{d}\theta \frac{r^2}{\ell} = \frac{a^2}{\ell}\int_0^\infty \frac{\mathrm{d}\theta}{\cosh^2\lambda\theta} = \frac{a^2}{\lambda\ell}\int_0^\infty \frac{\mathrm{d}x}{\cosh^2 x} = \frac{a^2}{\lambda\ell} = \frac{a^2}{\sqrt{C - a^2v^2}}.$$