Exercise 1. *Particle in a* $1/r^n$ *potential*

Consider a mass m moving on a circular orbit of radius r_0 under the influence of a central force whose potential is given by

$$V(r) = -\frac{km}{r^n} \tag{1}$$

where k is some constant. Show that if $n < 2$, the circular orbit is stable under small oscillations, *i.e.* the mass will oscillate about the circular orbit.

Solution. For a particle moving under the influence of a central force the Lagrangian of the system is given by

$$\mathcal{L}(r, \dot{r}, \dot{\phi}) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\right] - V(r). \tag{S.1}$$

For a circular orbit, where the magnitude of the radius does not change with time, the Lagrangian is simplified to

$$\mathcal{L}(r, \dot{\phi}) = \frac{1}{2}mr^2\dot{\phi}^2 - V(r). \tag{S.2}$$

We introduce the effective potential

$$V_{eff} = V(r) + \frac{\ell^2}{2mr^2}, \tag{S.3}$$

where $\ell = mr^2\dot{\phi}$ is the angular momentum of the particle, which is a conserved quantity, since the Euler-Lagrange equation for the coordinate ϕ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\phi}\right) = 0. \tag{S.4}$$

The total energy E of the system is conserved, because of explicit time independence of the quantity V_{eff}. Therefore we obtain:

$$\frac{\partial V_{eff}}{\partial r} = \frac{\partial E}{\partial r} = 0. \tag{S.5}$$

For a $1/r^n$ potential, the derivatives of the effective potential are

$$V_{eff} = -\frac{km}{r^n} + \frac{\ell^2}{2mr^2} \tag{S.6}$$

$$\Rightarrow \quad \frac{\partial V_{eff}}{\partial r} = \frac{kmn}{r^{n+1}} - \frac{\ell^2}{mr^3} \tag{S.7}$$

$$\Rightarrow \quad \frac{\partial^2 V_{eff}}{\partial r^2} = -\frac{kmn(n+1)}{r^{n+2}} - \frac{3\ell^2}{mr^4}. \tag{S.8}$$

The condition for circular orbit, as shown before, is $\frac{\partial V_{eff}}{\partial r} = 0$. Plugging this into equation (S.7), we obtain

$$kmn = \frac{\ell^2 r^{n-2}}{m}. \tag{S.9}$$

The orbit is stable if

$$\left.\frac{\partial^2 V_{eff}}{\partial r^2}\right|_{r_0} > 0, \tag{S.10}$$

which amounts to say that r_0 is a minimum of the energy E. In our case we can insert the result from equation (S.9) into equation (S.8) to obtain a condition for stability:

$$3 - (n+1) > 0 \quad \Rightarrow \quad n < 2. \tag{S.11}$$

Exercise 2. ***The Central Drill***

Consider a satellite m orbiting around a planet of mass M, whose positions are $\vec{r}_1$ and $\vec{r}_2$ respectively. Let $\vec{R}$ be the position of the centre of mass of the planet-satellite system with $R \equiv |\vec{R}|$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$ the relative coordinate with $r \equiv |\vec{r}|$. Also, let $k = GMm$, where G is Newton's gravitational constant.

(i) Making use of Newtonian mechanics show that the equations of motion are given by:

$$\mu \ddot{\vec{r}} = -k \frac{\vec{r}}{r^3} \quad \text{and} \quad \dot{\vec{R}} = \text{constant}$$

Find μ, the *reduced mass* of the system.

Hint: The centre of mass of a N body system is given by:

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

where m_i and $\vec{r}_i$ are the masses and positions of the bodies.

Solution. From Newton's second law one obtains:

$$m\ddot{\vec{r}}_1 = -\frac{\partial V(r)}{\partial \vec{r}_1} \tag{S.12}$$

$$M\ddot{\vec{r}}_2 = -\frac{\partial V(r)}{\partial \vec{r}_2} \tag{S.13}$$

and from the third (or equivalently from the chain rule)

$$\frac{\partial V(|\vec{r}_1 - \vec{r}_2|)}{\partial \vec{r}_2} = -\frac{\partial V(|\vec{r}_1 - \vec{r}_2|)}{\partial \vec{r}_1}, \tag{S.14}$$

so that the equations of motions become

$$m\ddot{\vec{r}}_1 = -\frac{\partial V(r)}{\partial \vec{r}_1} \tag{S.15}$$

$$M\ddot{\vec{r}}_2 = +\frac{\partial V(r)}{\partial \vec{r}_1}. \tag{S.16}$$

By *adding* these equations and using the definition of the center of mass we find

$$m\ddot{\vec{r}}_1 + M\ddot{\vec{r}}_2 = 0 \tag{S.17}$$

$$\Longleftrightarrow \quad \frac{m\ddot{\vec{r}}_1 + M\ddot{\vec{r}}_2}{m+M} = 0 \tag{S.18}$$

$$\Rightarrow \quad \ddot{\vec{R}} = 0 \tag{S.19}$$

$$\Rightarrow \quad \dot{\vec{R}} = \text{constant}. \tag{S.20}$$

On the other hand, by *subtracting* the equations of motions of the two bodies, we obtain:

$$\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = -\frac{1}{m}\frac{\partial V(r)}{\partial \vec{r}_1} - \frac{1}{M}\frac{\partial V(r)}{\partial \vec{r}_1} = -\frac{m+M}{mM}\frac{\partial V(r)}{\partial \vec{r}_1} \tag{S.21}$$

$$\Longleftrightarrow \quad \frac{mM}{m+M}\ddot{\vec{r}} = -\frac{\partial V(r)}{\partial \vec{r}} \tag{S.22}$$

$$\Rightarrow \quad \mu\ddot{\vec{r}} = -k\frac{\vec{r}}{r^3} \tag{S.23}$$

where we introduced the reduced mass $\mu \equiv \frac{mM}{m+M}$ and we performed the derivation of the gravitational potential $V(r) = -\frac{k}{r}$.

(ii) The Lagrangian of the system has been obtained in the lecture and reads

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r), \tag{2}$$

where r, θ are polar coordinates in the relative reference frame. Rotate your system by a little amount:

$$\theta \to \theta + \epsilon$$

and use Noether's theorem or a careful observation of the Lagrangian to show that the angular momentum is conserved:

$$\mu r^2\dot{\theta} = \text{constant} = M.$$

Solution. Consider the change of the Lagrangian with respect to the transformation parameter:

$$\frac{\delta L}{\delta\epsilon} = \frac{\partial L}{\partial\theta}\frac{\partial\theta}{\partial\epsilon} + \frac{\partial L}{\partial\dot{\theta}}\frac{\partial\dot{\theta}}{\partial\epsilon} = \left(\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}\right)\frac{\partial\theta}{\partial\epsilon} + \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\frac{\partial\theta}{\partial\epsilon}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\frac{\partial\theta}{\partial\epsilon}\right)$$

and note that the Lagrangian remains *unchanged* by this transform, i.e.:

$$\frac{\delta L}{\delta\epsilon} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\frac{\partial\theta}{\partial\epsilon}\right)$$

This is the statement of Noether's theorem for our case. It yields:

$$\frac{\partial L}{\partial\dot{\theta}}\frac{\partial\theta}{\partial\epsilon} = \frac{\partial L}{\partial\dot{\theta}} = const. = \mu r^2\dot{\theta}$$

The masses are constant and therefore:

$$\mu r^2\dot{\theta} = const.$$

The $\vec{z}$ vector remains unchanged by the transformation (we formally transformed by $\delta\vec{r} = \epsilon\hat{z}\times\vec{r}$) and therefore:

$$\mu r^2\dot{\theta}\hat{z} = const. = \vec{M}$$

(iii) About five centuries ago, Kepler figured out that the area swept by r in a given time is constant. Can you use conservation of angular momentum to conclude the same in 2015?

Solution. Observe that $\frac{dA}{dt} = \frac{1}{2}r(r\frac{d\theta}{dt})$. It then follows from conservation of angular momentum that this quantity is a constant.

(iv) Verify that by taking the time derivative of the total energy you can recover the equation of motion in r.

Solution. In terms of r the total energy of the system is given as

$$E = \frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{M^2}{2\mu r^2}. \tag{S.24}$$

Deriving this expression with respect to the time and using the chain rule we first obtain:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mu\dot{r}\ddot{r} + \frac{\partial V}{\partial r}\dot{r} - \frac{M^2}{\mu r^3}\dot{r}. \tag{S.25}$$

Since the energy of the problem is conserved, we can set $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$ and get

$$\mu\ddot{r} + \frac{\partial V}{\partial r} - \frac{M^2}{\mu r^3} = 0 \tag{S.26}$$

By substituting the angular momentum in favour of the angle we recover

$$\mu\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial V}{\partial r} \tag{S.27}$$

which is the equation of motion for the magnitude of the relative vector r stemming from

$$\mu\ddot{\vec{r}} = -k\frac{\vec{r}}{r^3}. \tag{S.28}$$

Exercise 3. *Symmetry of the orbit*

In the lecture we have seen that the two body problem can be reduced to a single body problem with reduced mass μ and a Lagrangian given by

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r). \tag{3}$$

(a) Using conservation of angular momentum L, show that the equation of motion for the radius can be expressed as:

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} = -\frac{\partial V}{\partial r} \tag{4}$$

(b) Starting from conservation of angular momentum, convert equation (4) for $r(t)$ into the following equation for $r(\theta)$:

$$\frac{L^2}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) - \frac{L^2}{\mu r^3} = f(r), \tag{5}$$

where $f(r)$ is the conservative force due to the potential $V(r)$.

(c) Using the substitution $u = \frac{1}{r}$, show that the differential equation for the orbit satisfies $u(\theta) = u(-\theta)$. What does this imply for a practical construction of the orbit?

Solution.

(a) The conservation of angular momentum L is contained in the Euler-Lagrange equation for the angular variable θ:

$$\mu r^2\dot{\theta} = L, \tag{S.29}$$

where μ is the reduced mass. The equation of motion for the radius r can then be obtained from the Euler-Lagrange equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{\partial\mathcal{L}}{\partial r} \tag{S.30}$$

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{\partial V}{\partial r} \tag{S.31}$$

Upon inserting $\dot{\theta} = \frac{L}{\mu r^2}$ in the equation of motion we readily obtain:

$$\mu\ddot{r} - \frac{L^2}{\mu r^3} = -\frac{\partial V}{\partial r} \tag{S.32}$$

(b) The conservation of angular momentum can be rewritten in a differential form:

$$L\mathrm{d}t = \mu r^2\mathrm{d}\theta. \tag{S.33}$$

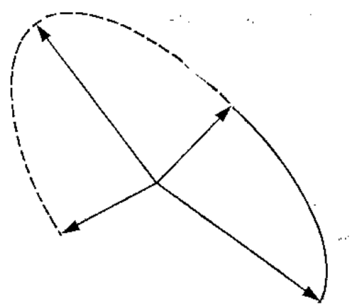

Figure 1: Illustration of the construction of the full orbit for the two-body problem from inversions with respect to the apsidal vectors for the case of an ellipse. Image taken from H. Goldstein, *Classical Mechanics*, Addison-Wesley (1951).

From this relation we obtain the time derivatives in terms of θ as:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{L}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta} \tag{S.34}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{L}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{L}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\right) \tag{S.35}$$

and inserting the second time derivative into the equation of motion we obtain

$$\frac{L}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{L}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) - \frac{L^2}{\mu r^3} = f(r), \tag{S.36}$$

where we also used the fact that $f(r) = -\frac{\partial V}{\partial r}$.

(c) After setting $u(\theta) = \frac{1}{r(\theta)}$, we first note that

$$\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{\mathrm{d}(1/r)}{\mathrm{d}\theta} = -\frac{\mathrm{d}u}{\mathrm{d}\theta}, \tag{S.37}$$

which means that the first term of equation (S.36) is written in terms of the inverse distance u as

$$-\frac{L^2}{\mu}u^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) - \frac{L^2}{\mu}u^3 = f\left(\frac{1}{u}\right) \tag{S.38}$$

$$\frac{L^2u^2}{\mu}\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u\right) = -f\left(\frac{1}{u}\right). \tag{S.39}$$

$$u'' + u = -\frac{\mu}{L^2}\frac{f(1/u)}{u^2} = -\frac{\mu}{L^2}\frac{\mathrm{d}}{\mathrm{d}u}V\left(\frac{1}{u}\right). \tag{S.40}$$

Now we can perform an hypothetical coordinate transformation $\theta \to -\theta$, which is nothing but an inversion with respect to the symmetry axes of the orbit. The differential equation (S.39) is manifestly invariant with respect to this angle transformation, since only the second derivative with respect of θ appears in the equation. The initial conditions

$$u_0 = u(0), \qquad \left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)_{\theta=0} = 0 \tag{S.41}$$

will also stay invariant. Therefore, the equation for the orbit will be the same irrespective of whether it is expressed with the variable θ or with $-\theta$: the orbit is symmetric with respect to the apsidal vectors. This implies that for a bound orbit (a circle or an ellipse), we only need to know a quarter of the orbit (from $\theta = 0$ to $\theta = \pi/2$) to obtain the full orbit (see sketch). The remaining parts can be reconstructed from reflections with respect to the apsidal vectors. For open orbits (parabolae and hyperbolae), there is only one symmetry axis and the orbit must be known for the interval $\theta \in [0, \pi]$.

Exercise 4. *Virial mass of clusters*

Among the applications of the virial theorem, there is an important one in astrophysics. Consider an isotropic globular cluster of stars or galaxies, modelled as a uniform distribution of point-like masses m_i located at positions $\vec{r}_i$. Use the virial theorem to prove that, if one is able to measure the size and the distribution of velocities of the cluster, its mass M can be obtained as

$$M = \frac{5}{3}\frac{R\langle v^2\rangle}{G}, \tag{6}$$

where R is the radius of the cluster and $\langle v^2\rangle$ an appropriate mean square velocity. Typically, it is not possible to directly observe the velocities of the single stars or galaxies, which have to be inferred from Doppler shifts. What is actually measured is thus only the distribution of velocities along the line of sight. Re-express $\langle v^2\rangle$ in terms of the mean square Doppler velocity σ^2, and substitute it into the formula for M.

Solution. For a spherical collection of isotropic objects bound by a gravitational potential

$$U(\vec{r}_i) = -\sum_{i<j} G\frac{m_i m_j}{|\vec{r}_i - \vec{r}_j|}, \tag{S.42}$$

the total gravitational binding energy is obtained by integrating over the full radius R the energy of infinitesimal shells of radius dr and mass dM. Since we assumed that the cluster is uniform, we can work with the density $\rho = \frac{M}{\frac{4}{3}\pi R^3}$ to obtain:

$$U = \int \mathrm{d}U = -\int G\frac{M(r)\mathrm{d}M}{r} = -\int_0^R G\frac{\left(\frac{4}{3}\pi\rho r^3\right)\left(4\pi r^2\rho\mathrm{d}r\right)}{r} =$$
$$= -\frac{16}{3}\pi^2 G\rho^2\int_0^R r^4\mathrm{d}r = -\frac{16}{15}\pi^2 G\rho^2 R^5 = -\frac{3GM^2}{5R}, \tag{S.43}$$

where in the last step we used $M = M(R) = \frac{4}{3}\pi R^3\rho$ for the total mass of the cluster. Upon taking the time average we therefore obtain:

$$\langle U\rangle = -\frac{3}{5}\frac{GM^2}{R}. \tag{S.44}$$

for the total gravitational energy.[1] The kinetic energy of the cluster is given by

$$T = \frac{1}{2}\sum_i m_i v_i^2, \tag{S.45}$$

but neither m_i nor v_i is an astronomically measurable quantity. Therefore we rewrite the kinetic energy as

$$T = \frac{M}{2}\sum_i \frac{m_i}{M}v_i^2 \equiv \frac{M}{2}\langle v^2\rangle \tag{S.46}$$

where $\langle v^2\rangle$ is a (weighted) average over all masses of the system. Using the virial theorem we can relate T and U and obtain a formula for M:

$$2\langle T\rangle = -\langle U\rangle \tag{S.47}$$
$$\Rightarrow \quad M\langle v^2\rangle = \frac{3}{5}\frac{GM^2}{R} \tag{S.48}$$
$$\Rightarrow \quad M = 5\frac{R\langle v^2\rangle}{3G}. \tag{S.49}$$

For 3-dimensional, isotropic systems $\langle v^2\rangle$ is related to the mean square velocity along a single axis by

$$\langle v^2\rangle = \langle v_x^2 + v_y^2 + v_z^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle = 3\langle v_x^2\rangle. \tag{S.50}$$

If we take e.g. the x axis to be aligned with the line of sight from earth to the cluster, $\langle v_x^2\rangle$ coincides with the variance of the distribution of measured Doppler velocities σ^2. Thus the virial theorem gives:

$$M = 5\frac{R\sigma^2}{G}; \tag{S.51}$$

[1] In case the cluster is in dynamical motion, it might be necessary to replace R with its time average.

the value obtained by this procedure is sometimes called the “virial mass” of a cluster.

Remark: A very important result can be obtained by computing the virial mass of galaxy clusters. For the Coma galaxy cluster, for instance, taking $R = 10^6$pc and $\sigma^2 = 1.5 \cdot 10^3$ km/s, one finds a value of $M = 10^{15} M_\odot$ (1000 trillions of solar masses). By comparing this with the mass of the visible stars in the cluster $M_S = 10^{13} M_\odot$, Fritz Zwicky realised in 1933 that there had to be roughly a factor 100 more mass not emitting light. This astronomical observation was one of the very first evidences pointing towards the existence of what Zwicky christened simply “dark matter”.