Exercise 1. ***Pendulum large oscillations***

Given a simple pendulum of mass m and length l. Assume that, at the initial time, the pendulum has zero velocity and it forms an angle ϕ_0 with respect to the vertical axis. Do not assume, for the moment, small oscillations.

a) Determine the period of oscillation as a function of its amplitude.

It is not required to explicitly perform the integral: you should be able to express it as a *complete elliptic integral of the first kind*

$$K(k) \;=\; \int_0^{\frac{\pi}{2}} \frac{d\xi}{\sqrt{1-k^2\sin^2\xi}}\,. \tag{1}$$

Hint: use the fact that $(1-\cos\phi)/2 = \sin^2(\phi/2)$ *first and then the substitution* $\sin\xi = \sin\frac{\phi}{2}\,/\sin\frac{\phi_0}{2}$ *, where* ϕ *is the angle the pendulum forms with the vertical axis.*
What is the difference between this case and the case of small oscillations seen in the third exercise sheet?

b) Knowing that the integral defined above has the following Taylor expansion

$$K(k) \;=\; \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2\,n)!}{2^{2n}\,(n!)^2}\right]^2 k^{2n}\,, \tag{2}$$

express the period of the pendulum as a power expansion up to the fourth power of the amplitude. Do you recognise any of the terms appearing?

Solution.

a) The conservation of energy of the pendulum reads

$$E \;=\; \frac{1}{2}m\,l^2\,\dot\phi^2 \;-\; m\,g\,l\,\cos\phi \;=\; -mgl\,\cos\phi_0, \tag{S.1}$$

where the angle ϕ is the angle between the string and the vertical, and ϕ_0 the maximum value of ϕ. Calculating the period as the time required to go from $\phi = 0$ to $\phi = \phi_0$ multiplied by four, we find

$$T \;=\; 4\sqrt{\frac{l}{2\,g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\cos\phi\,-\,\cos\phi_0}} \;=\; 2\sqrt{\frac{l}{g}}\int_0^{\phi_0}\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2}\,-\,\sin^2\frac{\phi}{2}}} \tag{S.2}$$

The substitution $\sin\xi = \sin\frac{\phi}{2}\,/\sin\frac{\phi_0}{2}$ means that

$$\cos\xi\,d\xi \;=\; \frac{1}{2}\frac{\cos\frac{\phi}{2}}{\sin\frac{\phi_0}{2}}\,d\phi \tag{S.3}$$

and the integrand becomes

$$\frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2}\,-\,\sin^2\frac{\phi}{2}}} = \frac{d\phi}{\sin\frac{\phi_0}{2}\sqrt{1-\sin^2\xi}} = \frac{d\phi}{\sin\frac{\phi_0}{2}\,\cos\xi} \tag{S.4}$$

$$(\text{S.3}) \rightarrow = \frac{2\,d\xi}{\cos\frac{\phi}{2}} = \frac{2\,d\xi}{\sqrt{1-\sin^2\frac{\phi}{2}}} = \frac{2\,d\xi}{\sqrt{1-\sin^2\frac{\phi_0}{2}\,\sin^2\xi}} \tag{S.5}$$

convert this to $T = 4\sqrt{l/g}\,K(\sin\frac{\phi_0}{2})$

b) For small oscillations

$$\sin\frac{\phi_0}{2} \simeq \frac{\phi_0}{2} - \frac{\phi_0^3}{2^3\,3!} + \dots \tag{S.6}$$

and expansion of the function K gives

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{2}{4}\right)^2 k^2 + \left(\frac{4!}{16\cdot 4}\right)^2 k^4 + \dots\right] = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \dots\right] \tag{S.7}$$

$$T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\phi_0^2 + \frac{11}{3\cdot 2^{10}}\phi_0^4 + \dots\right) \tag{S.8}$$

The first term is the result which is obtained by requiring small oscillations since the beginning. Here we have obtained a correction to this first approximation. Including higher order corrections we would get a better approximation, and by including all the terms we would obtain the exact result, valid also when oscillations aren't small.

Exercise 2. *Rotating bead on a plane*

A bead of mass m is constraint to move on a horizontal disc placed at height H from the ground. Now we puncture a small hole at the center of the disc, we attach a massless wire of length L to the bead and we pass it through the hole; when the bead is in the rest position at the center of the disc, then, the wire completely hangs loose vertically. In order to describe the motion of bead use cylindrical coordinates (ρ, ϕ). We look at the following two cases:

(a) We attach a weight of mass M and height $H - L$ to the other end of the wire, in a way that the weight touches the ground and the wire is completely straight above it. Now we pull the bead horizontally, such that the weight reaches a height of z from the ground, and we give a spin to the bead, placing it in an (instantaneous) circular motion with angular velocity ω (see Fig 1).

  (i) Derive the Lagrangian for the bead in this configuration. Neglect the friction in your derivation.

  (ii) Write down the Euler-Lagrange equations, and using one of the two show that the angular momentum l of the bead is a conserved quantity.

(b) Now we remove the weight attached to the wire and we replace it with a spring fixed on the ground, which has a spring constant k and rest position at $z_0 = H - L$. Similarly as before, we pull the bead horizontally until the spring is stretched to a height z, and then we let it spin on the disc with angular velocity ω (see Fig 2).

  (i) Derive the Lagrangian for the bead in this other configuration. Neglect the friction.

  (ii) Write down the Euler-Lagrange equations and show that the angular momentum l of the bead is conserved.

(c) In both cases cast the Euler-Lagrange equations into the following form:

$$m\ddot{\rho} = -\frac{d}{d\rho}\left(U_{\text{eff.}}\right), \tag{3}$$

where $U_{\text{eff.}}(\rho) = U(\rho) + \frac{1}{2}\frac{l^2}{m\rho^2}$. Show that for both cases there is an orbit which is restricted to some region $[\rho_{\min}, \rho_{\max}]$.

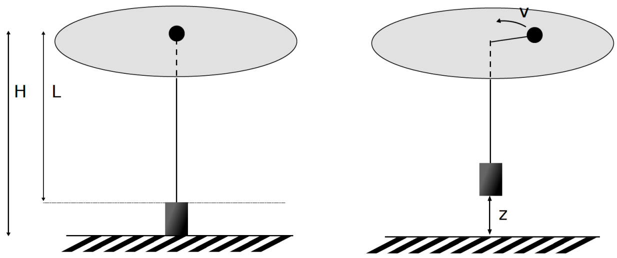

Figure 1: Graphical depiction of the system for exercise 2a).

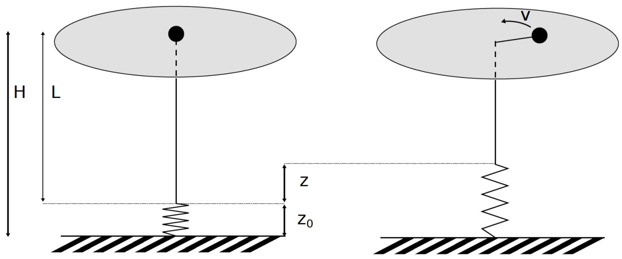

Figure 2: Graphical depiction of the system for exercise 2b).

Solution. This is a problem of a particle under the action of a central potential $U = U(\rho)$. The independent coordinates are taken to be the distance ρ from the origin and the angle ϕ with respect to the x axis. Then,

$$\begin{aligned} x &= \rho \cos \phi \,, \\ y &= \rho \sin \phi \,. \end{aligned} \tag{S.9}$$

The time derivative is

$$\begin{aligned} \dot{x} &= \dot{\rho} \cos \phi - \rho \sin \phi \; \dot{\phi} \,, \\ \dot{y} &= \dot{\rho} \sin \phi + \rho \cos \phi \; \dot{\phi} \,. \end{aligned} \tag{S.10}$$

The kinetic energy is

$$T = \frac{m}{2} \left(\dot{x}^2 + \dot{y}^2 \right) = \frac{m}{2} \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right) . \tag{S.11}$$

Note that for the first case we could in principle add the kinetic energy of the mass M: $T_M = \frac{M}{2} \dot{z}^2 = \frac{M}{2} \dot{\rho}^2$ (since $z = \rho$) as well, but this only redefines some parameters and the equations keep the same form.

The Lagrangian is

$$\mathcal{L} = \frac{m}{2} \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 \right) - U(\rho) \,. \tag{S.12}$$

and $U(\rho)$ depends on the configuration. In the first case it is a linear function $U_1(\rho) = Mg\rho \, (= Mgz)$, while in the second case it is quadratic in the argument, $U_2(\rho) = \frac{1}{2} k \rho^2$. Then the Lagrange equations are:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\rho}} - \frac{\partial \mathcal{L}}{\partial \rho} = m\ddot{\rho} - m\rho\dot{\phi}^2 + \frac{\partial U}{\partial \rho} = 0 \,, \tag{S.13}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt} \left(m \rho^2 \dot{\phi} \right) = 0 \,. \tag{S.14}$$

The second equation implies that for any form of central potential $U(\rho)$, the angular momentum $m\rho^2\dot{\phi}$ is conserved. Let's fix the value of angular momentum to l (note that from the initial conditions: $l = mz^2\omega$). Then, we can rewrite (S.13) as

$$m\ddot{\rho} = -\frac{d}{d\rho} \left(U(\rho) + \frac{1}{2} \frac{l^2}{m\rho^2} \right) = -\frac{d}{d\rho} U_{\text{eff.}}(\rho). \tag{S.15}$$

Hence we have $U_{\text{eff.}}(\rho) = U(\rho) + \frac{1}{2} \frac{l^2}{m\rho^2}$.

Note that, if we multiply the last equation by $\dot\rho$, we obtain

$$m\ddot\rho\dot\rho = \frac{d}{dt}\left(\frac{1}{2}m\dot\rho^2\right) = -\frac{d}{dt}\left(U(\rho) + \frac{1}{2}\frac{l^2}{m\rho^2}\right), \tag{S.16}$$

which gives another first integral of motion, namely the total energy of the system,

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot\rho^2 + U(\rho) + \frac{1}{2}\frac{l^2}{m\rho^2}\right) = 0. \tag{S.17}$$

From (S.15) we see that the distance form the center varies in the same way as a coordinate ρ varies in the one-dimensional problem with an effective potential $U_{\text{eff.}}(\rho)$. We know that the motion of a point mass is restricted in the region where $U_{\text{eff.}}(\rho) \leq E$. Since for both cases $U_{\text{eff.}}$ has a local minimum $0 < \rho_0 < \infty$, the last inequality gives an annular region on the disk where the motion takes place: $\rho \in [\rho_{\min}, \rho_{\max}]$.

Exercise 3. *The variation of the energy of a holonomic system*

a) Write the kinetic energy

$$T = \frac{1}{2}\sum_{j=1}^{N} m_j(\dot{\vec r}_j)^2 \tag{4}$$

of a holonomic system of N particles as a function of n independent generalized coordinates q_i $(i = 1, \ldots, n)$ and show that for stationary constraints (i.e., constraints without explicit dependence on time)

$$f_k(\vec r_1, \vec r_2, \ldots, \vec r_j) = 0\,, \qquad k = 1, \ldots, d \quad \text{where } d \text{ is the number of constraints,} \tag{5}$$

T is a quadratic, homogeneous polynomial of the generalized velocities $\dot q_i$.

Hint. Write $T = T_2 + T_1 + T_0$ where T_2 is quadratic in $\dot q_i$, T_1 is linear in $\dot q_i$ and T_0 is independent of $\dot q_i$.

Solution. We start from expressing the position vectors in term of the generalized coordinates (and the time)

$$\vec r_j = \vec r_j(q_1, q_2, \ldots, q_n, t) \tag{S.18}$$

and by differentiating one obtains

$$\begin{aligned} T &= \frac{1}{2}\sum_{j=1}^{N} m_j \left(\sum_{i=1}^{n}\frac{\partial \vec r_j}{\partial q_i}\dot q_i + \frac{\partial \vec r_j}{\partial t}\right)^2 \\ &= \frac{1}{2}\sum_{j=1}^{N} m_j \left(\sum_{i,k}\frac{\partial \vec r_j}{\partial q_i}\cdot\frac{\partial \vec r_j}{\partial q_k}\dot q_i\dot q_k + 2\sum_i \frac{\partial \vec r_j}{\partial q_i}\cdot\frac{\partial \vec r_j}{\partial t}\dot q_i + \frac{\partial \vec r_j}{\partial t}\cdot\frac{\partial \vec r_j}{\partial t}\right) \\ &= T_2 + T_1 + T_0\,, \end{aligned} \tag{S.19}$$

where we have set,

$$\begin{aligned} T_2 &= \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}\dot q_i\dot q_k\,, \quad \text{with} \quad a_{ik} = \sum_{j=1}^{N} m_j \frac{\partial \vec r_j}{\partial q_i}\cdot\frac{\partial \vec r_j}{\partial q_k}\,, \\ T_1 &= \sum_{i=1}^{n} b_i\dot q_i\,, \quad \text{with} \quad b_i = \sum_{j=1}^{N} m_j\frac{\partial \vec r_j}{\partial q_i}\cdot\frac{\partial \vec r_j}{\partial t}\,, \\ T_0 &= \frac{1}{2}\sum_{j=1}^{N} m_j\frac{\partial \vec r_j}{\partial t}\cdot\frac{\partial \vec r_j}{\partial t}\,. \end{aligned} \tag{S.20}$$

We know that for stationary constraints (5) the positions of the particles $\vec{r}_j$ do not depend explicitly on the time t in eq. (S.18). Thus,

$$\frac{\partial \vec{r}_j}{\partial t} = 0 \qquad \longrightarrow \qquad T_1 = T_0 = 0\,, \tag{S.21}$$

that is, for stationary constraints the kinetic energy is a homogeneous polynomial of the generalized velocities,

$$T = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}\dot{q}_i\dot{q}_k\,. \tag{S.22}$$

b) Given that the total energy of a holonomic system is $E = T + U$, where T is the kinetic energy and U is the potential energy, use the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} + \tilde{Q}_i \tag{6}$$

to show that

$$\frac{dE}{dt} = \frac{d}{dt}\left(T_1 + 2T_0\right) - \frac{\partial T}{\partial t} + \frac{\partial U}{\partial t} + \sum_i \tilde{Q}_i\dot{q}_i\,, \tag{7}$$

where $\tilde{Q}_i$ is a non-potential generalized force.

Hint. Make use of the Euler formula for a homogeneous polynomial $f = f(x_1,\ldots,x_k)$ of degree m,

$$\sum_{i=1}^{k}\frac{\partial f}{\partial x_i}x_i = mf\,. \tag{8}$$

Solution. We start from the kinetic energy:

$$\frac{dT}{dt} = \sum_i\left(\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right) + \frac{\partial T}{\partial t}\,. \tag{S.23}$$

The second term above can be written as,

$$\frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) - \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i\,, \tag{S.24}$$

such that

$$\begin{aligned}\frac{dT}{dt} &= \sum_i\left(\frac{\partial T}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{d}{dt}\left(\sum_i\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial T}{\partial t}\\ &= \sum_i\left(\frac{\partial U}{\partial q_i} - \tilde{Q}_i\right)\dot{q}_i + \frac{d}{dt}\left(\sum_i\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) + \frac{\partial T}{\partial t}\,,\end{aligned} \tag{S.25}$$

where in the second line we used the Lagrange equations (6).

To continue, we use Euler formula for a homogeneous polynomial (8). Because the kinetic energy is written in terms of independent coordinates as in eq. (S.20), we apply Euler formula and write,

$$\begin{aligned}\sum_i\frac{\partial T_2}{\partial \dot{q}_i}\dot{q}_i &= 2T_2\,,\\ \sum_i\frac{\partial T_1}{\partial \dot{q}_i}\dot{q}_i &= T_1\,,\end{aligned} \tag{S.26}$$

Then the second term in the time derivative of the kinetic energy (S.25) is

$$\frac{d}{dt}\left(\sum_i\frac{\partial T}{\partial \dot{q}_i}\dot{q}_i\right) = \frac{d}{dt}\left(2T_2 + T_1\right) = \frac{d}{dt}\left(2T - T_1 - 2T_0\right)\,. \tag{S.27}$$

Substituting eq. (S.27) into eq. (S.25), we obtain the time derivative of the kinetic energy,

$$\frac{dT}{dt} = \frac{d}{dt}(T_1 + 2T_0) - \frac{\partial T}{\partial t} - \sum_i \left(\frac{\partial U}{\partial q_i} - \tilde{Q}_i \right) \dot{q}_i \,. \tag{S.28}$$

The time derivative of the potential energy is,

$$\frac{dU}{dt} = \sum_i \frac{\partial U}{\partial q_i} \dot{q}_i + \frac{\partial U}{\partial t} \,. \tag{S.29}$$

Finally, summing eqs. (S.28) and (S.29), we obtain the time derivative of the total energy of a holonomic system (7).

c) Consider a few cases of the result (7).

(i) system with stationary constraint

(ii) the potential energy does not depend explicitly on time and (i)

(iii) a conservative system and (ii)

Solution.

(i) For stationary constraints,

$$T_1 = T_0 = 0 \,, \qquad \frac{\partial T}{\partial t} = 0 \,, \tag{S.30}$$

so the time derivative of the total energy becomes,

$$\frac{dE}{dt} = \frac{\partial U}{\partial t} + \sum_i \tilde{Q}_i \dot{q}_i \,. \tag{S.31}$$

(ii) If furthermore the potential energy does not depend explicitly on time,

$$\frac{dE}{dt} = \sum_i \tilde{Q}_i \dot{q}_i \,. \tag{S.32}$$

(iii) A system for which the energy is conserved is called a *conservative system*, and has:

* stationary constraints,
* the potential energy which does not depend explicitly on time,
* no non-potential forces.

d) Non-potential forces $\tilde{Q}_i$ are called gyroscopic if

$$\sum_i \tilde{Q}_i \dot{q}_i = 0 \,. \tag{9}$$

Consider a generalized force which depends linearly on the generalized velocities

$$\tilde{Q}_i = \sum_j a_{ij} \dot{q}_j \,. \tag{10}$$

Under what conditions is $\tilde{Q}_i$ gyroscopic? Is the *Coriolis force* acting on a particle of mass m a gyroscopic force?

$$\vec{\tilde{Q}} = -2m\,\vec{\omega} \times \vec{v} \tag{11}$$

Solution. The condition (9) is fulfilled if a_{ij} either vanishes or is antisymmetric. The Coriolis force is gyroscopic, since (in components) a_{ij} is antisymmetric,

$$\begin{pmatrix} \tilde{Q}_x \\ \tilde{Q}_y \\ \tilde{Q}_z \end{pmatrix} = -2m \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \,. \tag{S.33}$$