Exercise 1. ***The Lie Algebra of*** $SO(3)$

a) Consider the rotation of a vector around the axis $\hat{n}$ by an angle $\delta\phi$:

$$\vec{r} \to \vec{r} + \delta\vec{r} + \mathcal{O}(\delta\phi^2) \tag{1}$$

where $\delta\vec{r}$ can be expressed as:

$$\delta\vec{r} = \delta\vec{\phi} \times \vec{r} \tag{2}$$

with

$$\delta\vec{\phi} = \hat{n}\delta\phi \tag{3}$$

Starting from Equation (2) with using Equation (3) find the generators of $SO(3)$, the group of rotations.

Solution. Plugging in the Equation (3) into Equation (2) we have:

$$(\hat{n} \times \vec{r})_i = \epsilon_{ijk}\hat{n}_j r_k \tag{S.1}$$

Then the generators are:

$$(J_i)_{jk} = i\epsilon_{ijk} \tag{S.2}$$

$$J_x = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_y = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_z = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{S.3}$$

b) Compute the commutators of the generators and find the Lie algebra. Determine the structure constants and confirm that these also obey the Lie algebra.

Solution.

$$[J_x, J_y] = -iJ_z \tag{S.4}$$

$$[J_y, J_z] = -iJ_x \tag{S.5}$$

$$[J_z, J_x] = -iJ_y \tag{S.6}$$

which can also be written as:

$$[J_i, J_j] = -i\epsilon_{ijk}J_k \tag{S.7}$$

Then the structure constants of $so(3)$ are:

$$f_{ijk} = -i\epsilon_{ijk} \tag{S.8}$$

c) The generators can be written as the derivatives of the representation matrices with respect to the small transformation parameters:

$$(J_i)_{jk} = \frac{1}{i}\frac{\partial(R_i)_{jk}(\phi)}{\partial\phi}|_{\phi=0} \tag{4}$$

where R_i is the rotation matrix for a rotation about a generic i axis. Exponentiate the generators to find the representation matrices:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\delta\phi & -\sin\delta\phi \\ 0 & \sin\delta\phi & \cos\delta\phi \end{pmatrix}, \quad R_y(\phi) = \begin{pmatrix} \cos\delta\phi & 0 & \sin\delta\phi \\ 0 & 1 & 0 \\ -\sin\delta\phi & 0 & \cos\delta\phi \end{pmatrix} \tag{5}$$

$$R_z(\phi) = \begin{pmatrix} \cos\delta\phi & -\sin\delta\phi & 0 \\ \sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

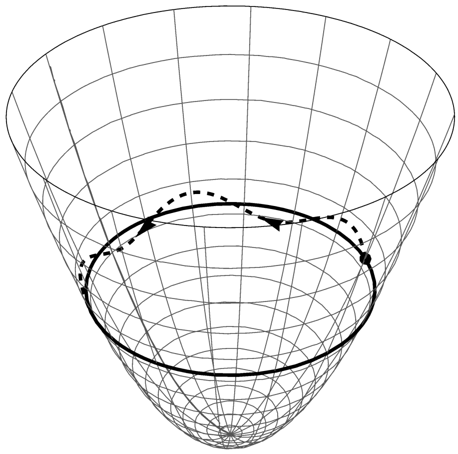

Figure 1: Exercise 2: point moving on a paraboloid.

Solution. The representation matrix expanded in the small transformation parameter can be written as:

$$(R_i)_{jk}\,(\delta\phi) = 1 + \delta\phi \frac{\partial (R_i)_{jk}(\phi)}{\partial \phi}|_{\phi=0} + (O)\left(\phi^2\right) \tag{S.9}$$

Ignoring the higher order terms:

$$(R_i)_{jk}\,(\delta\phi) \sim 1 + i\delta\phi\,(J_i)_{jk} \tag{S.10}$$

Now imagine we have two rotations around the same axis:

$$\mathbf{R}\,(\phi_1)\,\mathbf{R}\,(\phi_2) = \mathbf{R}\,(\phi_1 + \phi_2) \tag{S.11}$$

Assuming that $\phi_2 = \phi$ and $\phi_1 = \delta\phi$ is infinitesimal,

$$(1 + i\delta\phi\, J_i)\, R_i\,(\phi) \sim R_i\,(\phi + \delta\phi) \tag{S.12}$$

which gives:

$$\frac{d}{d\phi} R_i\,(\phi) = iJ_i R_i\,(\phi)$$

with a solution

$$R_i\,(\phi) = e^{i\phi J_i}$$

Exercise 2. *Point moving on a paraboloid*

A point particle of mass m, subject to gravity, moves on a smooth paraboloid surface with equation $z = c^2(x^2 + y^2)$.

a) Write down the lagrangian for this system using cylindrical coordinates (r, ϕ, z) and expressing the constraint through Lagrange multipliers.
b) Use the rotational symmetry around the z-axis to work out the associated conserved quantity using Noether's theorem.
c) Write down the Euler-Lagrange equations and compare with point b). Deduce also that, for any fixed value of the conserved quantity, there is a value r_0 of the radial coordinate r that satisfies the remaining equation of motion with $\dot{r}(t) = 0$.
d) Expanding the equations around r_0 by means of $r(t) \equiv r_0 + \Delta r(t)$ with small Δr, find the motion of nearly circular orbits.

Solution.

a) The lagrangian is given by

$$L = T - U - \lambda F, \tag{S.13}$$

where T is the kinetic energy, U the gravitational potential and F the function expressing the constraint. One may start from cartesian coordinates or write down these terms in cylindrical coordinates directly; in any case the expressions read

$$\begin{cases} T = \frac{m}{2}(v_r^2 + v_\phi^2 + v_z^2) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2), \\ U = mgz, \\ F = z - c^2(x^2 + y^2) = z - c^2 r^2, \end{cases} \tag{S.14}$$

which gives

$$L[r, \dot{r}, \phi, \dot{\phi}, z, \dot{z}; \lambda] = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - mgz - \lambda(z - c^2 r^2). \tag{S.15}$$

It is interesting to compare this to the lagrangian where the constraint has been removed by an appropriate choice of generalized coordinates. For instance, eliminating z in favor of its expression through r, we find

$$L[r, \dot{r}, \phi, \dot{\phi}] = \frac{m}{2}[(1 + 4c^4 r^2)\dot{r}^2 + r^2\dot{\phi}^2] - mgc^2 r^2. \tag{S.16}$$

b) A rotation around the z axis in cylindrical coordinates corresponds to the transformation

$$\begin{cases} r \to r' = r, \\ \phi \to \phi' = \phi + \epsilon, \\ z \to z' = z, \end{cases} \tag{S.17}$$

for any constant $\epsilon \in \mathbb{R}$. The lagrangian, being explicitly independent of ϕ, is clearly invariant under (S.17). Thus Noether's theorem guarantees that

$$J = \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}q_i}{\mathrm{d}\epsilon}, \tag{S.18}$$

is a constant during motion. In this case clearly

$$\frac{\partial r'}{\partial \epsilon} = 0, \qquad \frac{\partial \phi'}{\partial \epsilon} = 1, \qquad \frac{\partial z'}{\partial \epsilon} = 0, \qquad \Rightarrow \qquad J = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}. \tag{S.19}$$

Therefore we find that the z component of the angular momentum is a conserved quantity.

c) The Euler-Lagrange equations read

$$\begin{cases} mr\dot{\phi}^2 + 2\lambda c^2 r - m\ddot{r} = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\phi}) = 0, \\ -mg - \lambda - m\ddot{z} = 0, \\ z - c^2 r^2 = 0. \end{cases} \tag{S.20}$$

Clearly the equation relative to ϕ gives directly the same result as (S.19); this was expected because $J = mr^2\dot{\phi}$ is the generalized momentum conjugate to ϕ and the lagrangian does not depend on ϕ directly. Three equations of motion can be immediately solved yielding

$$\begin{cases} r^2\dot{\phi} = J/m \equiv \ell, \\ \lambda = -m(g + \ddot{z}), \\ z = c^2 r^2. \end{cases} \tag{S.21}$$

After trivial steps the first equation becomes

$$\ell^2 - 2c^2 r^4[g + 2c^2(\dot{r}^2 + r\ddot{r})] - r^3\ddot{r} = 0; \tag{S.22}$$

thus clearly if $\dot{r}(t) = 0$ setting $r(t) = r_0$ gives[1]

$$\ell^2 = 2gc^2 r_0^4. \tag{S.23}$$

[1] After studying central potentials, you can try to re-obtain this result using the effective radial potential.

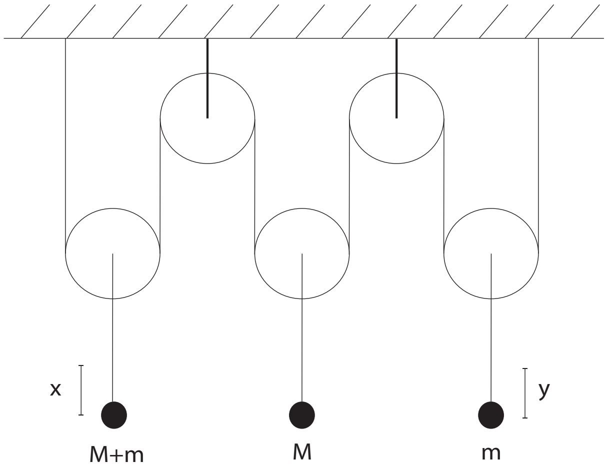

Figure 2: More sophisticated Atwood machine.

d) At first order in Δr, using the expansion

$$(1+\varepsilon)^n = 1 + n\varepsilon + \mathcal{O}(\varepsilon^2), \tag{S.24}$$

the equation becomes

$$\ell^2 - 2c^2(r_0^4 + 4r_0^3\Delta r)[g + 2c^2 r_0 \ddot{\Delta r}] - r_0^3\ddot{\Delta r} = 0, \tag{S.25}$$

$$2c^2(4gr_0^3\Delta r + 2c^2 r_0^5\ddot{\Delta r}) + r_0^3\ddot{\Delta r} = 0, \tag{S.26}$$

$$8gc^2\Delta r + (1 + 4c^4 r_0^2)\ddot{\Delta r} = 0. \tag{S.27}$$

A perturbation of the stable circular orbit for a given angular momentum J will therefore feature small harmonic radial oscillations with frequency

$$\omega = \sqrt{\frac{8gc^2}{1 + 4c^4 r_0^2}}. \tag{S.28}$$

Exercise 3. *Atwood Machine II*

Consider a more sophisticated Atwood machine as shown in Figure 2. Given the masses m, M and $m + M$ and the displacement coordinates x and y of the left and right masses as depicted, use Noether's theorem to derive the conserved momentum in this problem. Assuming that the system starts at rest, show that

$$\left(m^2 - 2M^2\right)\dot{x} = \left(M^2 + m^2\right)\dot{y}. \tag{7}$$

Solution. As the length of the string is fixed, displacement of the mass on the left by x and displacement of the mass on the right by y imply a displacement of $-x-y$ for the mass in the middle. The Lagrangian is then

$$\begin{aligned} L &= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}M(-\dot{x}-\dot{y})^2 + \frac{1}{2}m\dot{y}^2 - g((M+m)x - M(x+y) + my) \\ &= \frac{1}{2}(2M+m)\dot{x}^2 + M\dot{x}\dot{y} + \frac{1}{2}(M+m)\dot{y}^2 - g(mx + (m-M)y). \end{aligned}$$

We can see that this Lagrangian is invariant under the transformation

$$\begin{aligned} y &\mapsto y + \epsilon \\ x &\mapsto x + \frac{M-m}{m}\epsilon, \end{aligned}$$

and so Noether's theorem tells us that the conserved momentum is

$$\begin{aligned} P &= \frac{\partial L}{\partial \dot{x}} \frac{M-m}{m} + \frac{\partial L}{\partial \dot{y}} \\ &= ((2M+m)\dot{x} + M\dot{y}) \frac{M-m}{m} + M\dot{x} + (M+m)\dot{y} \\ &= \frac{2M^2 - m^2}{m}\dot{x} + \frac{M^2 + m^2}{m}\dot{y}. \end{aligned}$$

Assuming the masses to be initially at rest, $P = 0$ and so Equation 7 follows.