Exercise 1. ***Lagrange Multipliers and Generalised Forces***

Last time, you have seen that when the system is described by n coordinates $q_1, ..., q_n$ (e.g. $n = 2$ for the double pendulum), then we have to solve n Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i},\ i = 1, 2, ..., n \tag{1}$$

Suppose for simplicity that we describe position of a point mass by three Cartesian coordinates (r_1, r_2, r_3). Then (1) can be replaced simply by one vector equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) = \frac{\partial L}{\partial \mathbf{r}} \tag{2}$$

where $\frac{\partial}{\partial \mathbf{r}} = (\frac{\partial}{\partial r_1}, \frac{\partial}{\partial r_2}, \frac{\partial}{\partial r_3})$ is just a different notation for the gradient.

a) Show using this notation that for a Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r})$ the Euler-Lagrange equations imply $m\ddot{\mathbf{r}} = -\frac{\partial V}{\partial \mathbf{r}}$. Does this result look familiar?

Let's now add the constraint $f(\mathbf{r}) = 0$ to our system. Mathematics than tells us that all we have to do is to solve Euler-Lagrange equations for the modified Lagrangian

$$L'(\mathbf{r}, \dot{\mathbf{r}}, \lambda) = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r}) - \lambda f(\mathbf{r}) \tag{3}$$

Note that from now on the scalar λ must be treated as any other coordinate and in general, it is time-dependent! But does the new term in our Lagrangian have any physical meaning?

The Euler-Lagrange equations now become

$$m\ddot{\mathbf{r}} = -\frac{\partial V}{\partial \mathbf{r}} - \lambda\frac{\partial f}{\partial \mathbf{r}} \tag{4}$$

The terms on the right side can be then interpreted as force due to the potential V and some additional force that arises due to the constraints. The simple calculation above can be generalised for case with N masses with positions $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N$. Note that $i = 1, 2, ..., N$ here indexes the masses, not the vector components!

b) We will explore the case $N = 2$ in more details. Consider two point masses m, M joined by a light rod of length ℓ. It is convenient to write the constraint in the form $f(\mathbf{r}_1, \mathbf{r}_2) = \ell^2 - (\mathbf{r}_1 - \mathbf{r}_2)^2 = 0$. Explain carefully using the observations above that the magnitude of force due to the constraint on each mass is related to the Lagrange multiplier λ by $F = 2\lambda\ell$. What is the direction of the force?

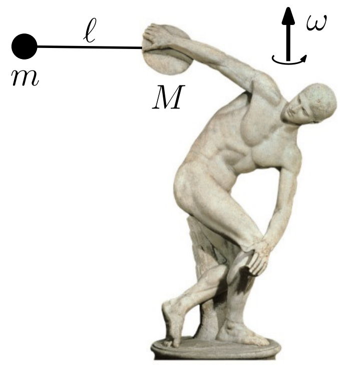
c) We now throw the system of masses to the free space where $V = 0$. Write down the Lagrangian and derive the equations of motion for each mass. Show that they imply that total momentum of the system is conserved.

d) Introduce the vector $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$ and show that the equations from c) imply

$$\mu\ddot{\mathbf{R}} = 2\lambda\mathbf{R} \text{ where } \mu = \frac{mM}{M + m} \tag{5}$$

e) Explain why the constraint implies $\mathbf{R}\cdot\dot{\mathbf{R}}=0$. By differentiating this further show that (5) can be rewritten as $-\mu\dot{\mathbf{R}}^2=2\lambda\ell^2$. Deduce that $\dot{\lambda}=0$.

f) Discobolus rotates with angular velocity $\boldsymbol{\omega}$ while holding the mass M in his right arm. He releases the system of masses at time $t=0$ when $\mathbf{R}(0)=\mathbf{R}_0$. Show that $\dot{\mathbf{R}}(0)=\boldsymbol{\omega}\times\mathbf{R}_0$ and use this to find λ and hence the force that each mass experiences in the subsequent motion. Can you show using simple Newtonian mechanics that your result is correct?



Solution.

a) Writing in components $\frac{\partial L}{\partial \dot{r}_i}=\frac{1}{2}m\frac{\partial}{\partial \dot{r}_i}(\dot{r}_j\dot{r}_j)=m\dot{r}_i$. Then EL eqns. are $m\ddot{r}_i=-\frac{\partial V}{\partial r_i}$, $i=1,2,3$ and back in vector notation $m\ddot{\mathbf{r}}=-\frac{\partial V}{\partial \mathbf{r}}$.

b) Force on the first mass is $\mathbf{F}_1=-\lambda\frac{\partial f}{\partial \mathbf{r}_1}=2\lambda(\mathbf{r}_1-\mathbf{r}_2)$. Using $|\mathbf{r}_1-\mathbf{r}_2|=\ell$ we find that the magnitude of the force is $F=2|\lambda|\ell$. The force is parallel to the vector $\mathbf{r}_1-\mathbf{r}_2$ connecting the masses and it should point out of the mass, so we expect $\lambda<0$.

c) $L=\frac{1}{2}m\dot{\mathbf{r}}_1^2+\frac{1}{2}M\dot{\mathbf{r}}_2^2-\lambda\left[\ell^2-(\mathbf{r}_1^2-\mathbf{r}_2^2)\right]$. Then EL eqns. give

$$m\ddot{\mathbf{r}}_1=2\lambda(\mathbf{r}_1-\mathbf{r}_2), \tag{S.1}$$
$$M\ddot{\mathbf{r}}_2=-2\lambda(\mathbf{r}_1-\mathbf{r}_2) \tag{S.2}$$

Summing both equations implies $\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{r}}_1+M\dot{\mathbf{r}}_2)=0$, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{p}_1+\mathbf{p}_2)=0$, which means that the total momentum does not change with time.

d) Dividing both equations by the mass and subtracting them gives the equation for $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$.

$$\ddot{\mathbf{R}}=\ddot{\mathbf{r}}_1-\ddot{\mathbf{r}}_2=2\lambda(\mathbf{r}_1-\mathbf{r}_2)\left(\frac{1}{m}+\frac{1}{M}\right)=2\lambda\mathbf{R}\frac{m+M}{mM}\Rightarrow\mu\ddot{\mathbf{R}}=2\lambda\mathbf{R} \tag{S.3}$$

e) The constraint is $\mathbf{R}^2=\ell^2$ where ℓ is constant. Differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{R}^2\right)=2\mathbf{R}\cdot\dot{\mathbf{R}}=0 \tag{S.4}$$

If we differentiate $\mathbf{R}\cdot\dot{\mathbf{R}}=0$ once more, we obtain $\mathbf{R}\cdot\ddot{\mathbf{R}}+\dot{\mathbf{R}}^2=0$. Now dotting the last equation in (S.3) with $\mathbf{R}$ gives $\mu\ddot{\mathbf{R}}\cdot\mathbf{R}=2\lambda\mathbf{R}^2$. Trading $\mathbf{R}\cdot\ddot{\mathbf{R}}$ for $-\dot{\mathbf{R}}^2$ and $\mathbf{R}^2$ for ℓ^2 gives the desired result

$$-\mu\dot{\mathbf{R}}^2=2\lambda\ell^2. \tag{S.5}$$

From this equation we obtain $\dot{\lambda}=-\frac{1}{\ell^2}\mu\dot{\mathbf{R}}\cdot\ddot{\mathbf{R}}$. Use again (S.3) to get

$$\dot{\lambda}=-\frac{2\lambda}{\ell^2}\mathbf{R}\cdot\dot{\mathbf{R}}=0 \tag{S.6}$$

In the last step we simply used the identity (S.4).

f) Describe Discobolus's arm by vector $\mathbf{a}$. At $t = 0$, the instantaneous velocities of masses m, M are $\dot{\mathbf{r}}_1(0) = \boldsymbol{\omega} \times (\mathbf{a} + \mathbf{R}_0)$, $\dot{\mathbf{r}}_2(0) = \boldsymbol{\omega} \times \mathbf{a}$, which implies $\dot{\mathbf{R}}(0) = \boldsymbol{\omega} \times \mathbf{R}_0$. Now using (S.5) and the time independence of λ we get

$$\lambda = \lambda(0) = -\frac{\mu}{2\ell^2}\dot{\mathbf{R}}(0)^2 = -\frac{\mu}{2\ell^2}\left[\omega^2\mathbf{R}_0^2 - (\boldsymbol{\omega}\cdot\mathbf{R}_0)^2\right] \tag{S.7}$$

We have used above the formula from ex. sheet 1. One may assume that the system of masses adjusts its orientation such that $\boldsymbol{\omega} \perp \mathbf{R}_0$ and so $\lambda = -\frac{1}{2}\mu\omega^2$. Then the force is given by formula from b) $F = 2|\lambda|\ell = \mu\omega^2\ell$.

Both masses rotate about their centre of mass with angular velocity ω. Radius of rotation of the mass m is then given by $d_1 = \frac{M}{m+M}\ell$ and so the force on this mass has magnitude $F = m\omega^2 d_1 = \mu\omega^2\ell$.

Exercise 2. *The Cycloidal slope*

Consider a slope with steepness described by the curve (cycloid)

$$\begin{cases} x = R\,(\theta - \sin\theta) \\ y = R\,(1 + \cos\theta) \end{cases} \qquad \text{with} \qquad 0 \le \theta \le \pi\,. \tag{6}$$

A ball is put on this slope at the point (x_0, y_0) identified by $\theta = \theta_0$ and starts rolling down (the ball can be approximated as point-like)

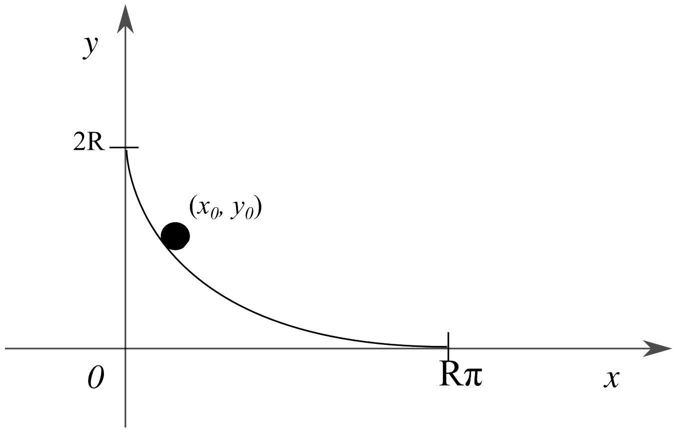


Figure 1: A ball rolling down the cycloidal slope described by Eq. (6).

a) Write the expression of the velocity of the ball depending on the position using the conservation of energy and use it to show that the time the ball takes to arrive at the lowest point is given by

$$T = \sqrt{\frac{R}{g}}\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}}\,d\theta\,. \tag{7}$$

b) How does the time T vary by changing the position of the point $(x_0\,,\,y_0)$ where the ball starts rolling?
Hint: Show that, for any $t_1 > 0$,

$$\int_0^{t_1}\frac{dt}{\sqrt{t\,(t_1 - t)}} = \pi\,. \tag{8}$$

Solution.

a) Conservation of energy reads

$$\frac{1}{2}\,m\,v^2 \;=\; mg(y_0 - y)\,, \tag{S.8}$$

from which it follows that the velocity of the ball is

$$v \;=\; \sqrt{2g(y_0 - y)}\,. \tag{S.9}$$

The time the ball takes to arrive at the bottom of the slop is then given by

$$\begin{aligned} T \;&=\; \int dt \;=\; \int \frac{dt}{ds}\,ds \;=\; \int \frac{ds}{v} \;=\; \int_{(x_0,y_0)}^{(R\pi,0)} \sqrt{\frac{dx^2+dy^2}{2\,g(y_0-y)}} \\ &=\; \int_{\theta_0}^{\pi} \sqrt{\frac{R^2\,(1-\cos\theta)^2 \;+\; R^2\,(-\sin\theta)^2}{2\,g\,(R\,\cos\theta_0 \;-\; R\,\cos\theta)}}\,d\theta \\ &=\; \sqrt{\frac{R}{g}}\,\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\,d\theta \end{aligned} \tag{S.10}$$

b) The time T is constant, i.e. it does not depend on the point the ball starts rolling from. This follows from the fact that

$$\int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\;d\theta \;=\; \pi\,. \tag{S.11}$$

Let us derive this result, solving the integral by performing a few changes of variables

$$\begin{aligned} \int_{\theta_0}^{\pi}\sqrt{\frac{1-\cos\theta}{\cos\theta_0-\cos\theta}}\;d\theta \;&=\; \int_{-1}^{t_0}\sqrt{\frac{1-t}{t_0-t}}\;\frac{dt}{\sqrt{1-t^2}} \;=\; \int_{-1}^{t_0}\frac{dt}{\sqrt{(1+t)\,(t_0-t)}} \\ =\; \int_0^{t_1}\frac{dt}{\sqrt{t\,(t_1-t)}} \;&=\; 2\int_0^{\sqrt{t_1}}\frac{dx}{\sqrt{t_1-x^2}} \;=\; 2\int_0^{\sqrt{t_1}}\frac{1}{\sqrt{1-(x/\sqrt{t_1})^2}}\,\frac{dx}{\sqrt{t_1}} \\ &=\; 2\int_0^1\frac{dy}{\sqrt{1-y^2}} \;=\; 2\,\arcsin y\Big|_0^1 \;=\; \pi \end{aligned} \tag{S.12}$$

we have used the following variable substitutions: $t = \cos\theta$, $x = \sqrt{t}$, $y = x/\sqrt{t_1}$ and $t_0 = \cos\theta_0$, $t_1 = t_0 + 1$

Exercise 3. *The Suspension Bridge*

Given an inextensible bridge of length l and mass m suspended freely under gravity between the points $A = (-D/2, 0)$ and $B = (D/2, 0)$ (Fig. 2). The goal of this exercise is to find the shape $y(x)$ of the bridge. Assume that the linear mass density $\mu = m/l$ of the bridge is constant and that $D < l$.

a) The bridge will take the shape that minimizes its potential energy. Find a contribution of the infinitesimal segment dl of the bridge to the overall potential energy V and show that the total potential energy is $\mu\mathrm{g}V[y]$, where g is the gravitational acceleration and

$$V[y] = \int_{-D/2}^{D/2} v(y, y')\mathrm{d}x = \int_{-D/2}^{D/2} y\sqrt{1+y'^2}\,\mathrm{d}x. \tag{9}$$

Fixed length l of the bridge imposes an integral constraint. Show that this constraint has the following form:

$$G[y] = \int_{-D/2}^{D/2} g(y, y')\mathrm{d}x = \int_{-D/2}^{D/2} \sqrt{1+y'^2}\mathrm{d}x = l. \tag{10}$$

Note that this is an example of a *non-holonomic constraint.*

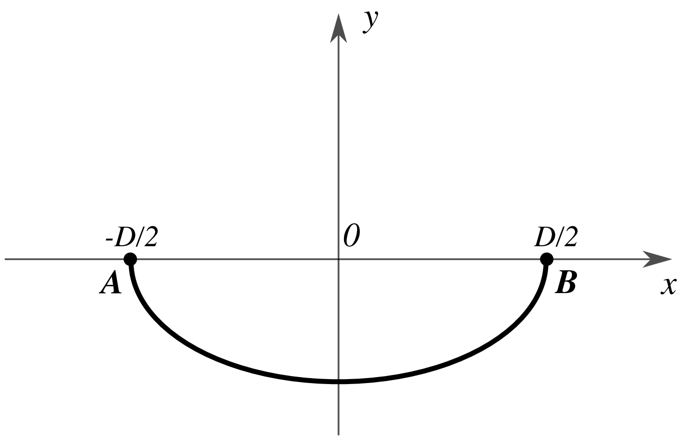


Figure 2: A bridge suspended freely between points A and B.

Bellow we want to cast the variational problem (9)-(10) with an integral constraint into an unconstrained variational problem.

b)** Define:

$$V^{\lambda}[y] = V[y] - \lambda G[y] = \int_{-D/2}^{D/2} F^{\lambda}(y, y')\,\mathrm{d}x \tag{11}$$

where λ is some constant.

Show that finding the extrema of the functional $V^{\lambda}[y]$ is equivalent to extremising the original functional $V[y]$ and satisfying the constraint $G[y] = l$ at the same time. *Even if you don't manage to prove this statement, use it and try to solve the rest of the problem.*

Hint. *Assume that $y = y(x)$ is a solution for the original variational problem (9)-(10) and embed it in a 2-parameter family $y = y(x; s, t)$, such that $y = y(x)$ correspondes to $(s, t) = (0, 0)$. Now you can treat $V[y]$ and $G[y]$ as functions of (s, t) and use the method of Lagrange multipliers.*

c) It has been shown previously that for systems with the integrand independent of the integration variable, E-L equations are equivalent to the Beltrami identity:

$$F^{\lambda}(y, y') - y' \frac{\partial F^{\lambda}(y, y')}{\partial y'} = C.$$

Show that for our case, this can be massaged to the following form:

$$\frac{y - \lambda}{\sqrt{1 + y'^2}} = C, \tag{12}$$

where C is a constant.

d) Solve the diff. equation (12) to find $y(x)$. Adjust two integration constants and λ such that your solution fulfills the boundary conditions and the constraint.

The shape you just (hopefully) found is called the *catenary*. You can now impress your peers by quoting that this is the shape of the suspension wires of the Golden Gate bridge of San Francisco and if turned upside down, the shape of the Gateway Arch in St. Louis.

Solution.

a) Easy!

b) After considering $y = y(x; s, t)$ family of functions as suggested in the hint, we have $V[y]$ as a function of (s, t). We also know that this function has an extremum at $(s, t) = (0, 0)$ subject to the condition that the point (s, t) lies of the curve $G[y] = l$. Hence we can use the methode of Lagrange multipliers and write the conditions of an extremum for the function $V^\lambda[y] = V[y] - \lambda(G[y] - l)$ in the extended space (s, t, λ):

$$\frac{\partial}{\partial s}V^\lambda[y] = 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial s}V[y] = \lambda\frac{\partial}{\partial s}G[y], \tag{S.13}$$

$$\frac{\partial}{\partial t}V^\lambda[y] = 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial t}V[y] = \lambda\frac{\partial}{\partial t}G[y], \tag{S.14}$$

$$\frac{\partial}{\partial \lambda}V^\lambda[y] = 0 \qquad \Rightarrow \qquad G[y] = l. \tag{S.15}$$

Note that the last equation is taking care of constraint. Moreover, we know that there exist a λ, such that (S.13)-(S.14) hold at the point $(s, t) = (0, 0)$.

We have

$$\frac{\partial}{\partial s}V[y] = \frac{\partial}{\partial s}\int_{-D/2}^{D/2} v(y(x; s, t), y'(x; s, t))\mathrm{d}x = \int_{-D/2}^{D/2}\left(\frac{\partial v}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial v}{\partial y'}\frac{\partial y'}{\partial s}\right)\mathrm{d}x = \int_{-D/2}^{D/2}\left(\frac{\partial v}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial v}{\partial y'}\right)\frac{\partial y}{\partial s}\mathrm{d}x.$$

In the last step we did integration by parts and dropped the boundary term due to the boundary conditions $y(-D/2, s, t) = y(D/2, s, t) = 0$ for any (s, t). Note that the boundary term has the form of $\frac{\partial v}{\partial y'}\frac{\partial}{\partial s}$ and vanishes even if the boundary values are non-zero, i.e. $y(-D/2, s, t) = A$ and $y(D/2, s, t) = B$.

Hence we can write (S.13)-(S.14) in the following form:

$$\int_{-D/2}^{D/2}\left(\frac{\partial v}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial v}{\partial y'}\right)\frac{\partial y}{\partial s}\mathrm{d}x = \lambda\int_{-D/2}^{D/2}\left(\frac{\partial g}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial y'}\right)\frac{\partial y}{\partial s}\mathrm{d}x, \tag{S.16}$$

$$\int_{-D/2}^{D/2}\left(\frac{\partial v}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial v}{\partial y'}\right)\frac{\partial y}{\partial t}\mathrm{d}x = \lambda\int_{-D/2}^{D/2}\left(\frac{\partial g}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial y'}\right)\frac{\partial y}{\partial t}\mathrm{d}x. \tag{S.17}$$

From these equations we see that the functional (variational) derivative of $V[y]$ in the direction of the vectors $\frac{\partial}{\partial s}y$ and $\frac{\partial}{\partial t}y$ from the space of functions are λ times the same derivative of $G[y]$. Since these two vectors give different directions in the space of functions, we can say that the full gradient of $V[y]$ is λ times the full gradient of $G[y]$. Thus we have

$$\int_{-D/2}^{D/2}\left[\left(\frac{\partial v}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial v}{\partial y'}\right) - \lambda\left(\frac{\partial g}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial y'}\right)\right](\delta y)\mathrm{d}x = 0, \tag{S.18}$$

for all δy with $\delta y(-D/2) = \delta y(-D/2) = 0$. Hence we get E-L for $F^\lambda(y, y') = v(y, y') - \lambda g(y, y')$:

$$\frac{\partial}{\partial y}(v - \lambda g) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}(v - \lambda g) = 0 \tag{S.19}$$

c) We can directly write the first integral of E-L (Beltrami identity):

$$\begin{aligned} F^\lambda - y'\frac{\partial F^\lambda}{\partial y'} &= \frac{y(1 + y'^2)}{\sqrt{1 + y'^2}} - \frac{yy'^2}{\sqrt{1 + y'^2}} + \lambda\left(\frac{y'^2}{\sqrt{1 + y'^2}} - \frac{1 + y'^2}{\sqrt{1 + y'^2}}\right) \\ &= \frac{y}{\sqrt{1 + y'^2}} - \lambda\frac{1}{\sqrt{1 + y'^2}} = C. \end{aligned}$$

Hence

$$\frac{y - \lambda}{\sqrt{1 + y'^2}} = C.$$

d) We can do change of variable $\tilde{y} = y - \lambda$. Then we have the following equation:

$$\frac{\tilde{y}}{\sqrt{1 + y'^2}} = C \qquad \Rightarrow \qquad \tilde{y}' = \pm\sqrt{\frac{\tilde{y}^2}{C^2} - 1}. \tag{S.20}$$

This differential equation should be familiar from the problem of soap films. The integration leads to:

$$\tilde{y} = C\cosh\left(\frac{x + C_1}{C}\right) = y - \lambda \qquad \Rightarrow \qquad y(x) = C\cosh\left(\frac{x + C_1}{C}\right) + \lambda.$$

Now we have two integration constants (C and C_1) and λ to fulfill the boundary conditions and the length constraint. The boundary conditions at A and B yield:

$$C\cosh\left(\frac{-D/2+C_1}{C}\right)+\lambda = C\cosh\left(\frac{D/2+C_1}{C}\right)+\lambda = 0$$

Since cosh is an even function, we have that $C_1 = 0$. Hence we have:

$$y(D/2) = C\cosh\left(\frac{D}{2C}\right)+\lambda = 0 \qquad \Rightarrow \qquad \lambda = -C\cosh\left(\frac{D}{2C}\right). \tag{S.21}$$

The integral constraint leads to:

$$l = 2\int_0^{D/2}\sqrt{1+\sinh^2\left(\frac{x}{C}\right)}\mathrm{d}x = 2\int_0^{D/2}\left|\cosh\left(\frac{x}{C}\right)\right|\mathrm{d}x = 2\int_0^{D/2}\cosh\left(\frac{x}{C}\right)\mathrm{d}x = 2C\sinh\left(\frac{D}{2C}\right).$$

Hence for the constant C we have the following equation

$$2C\sinh\left(\frac{D}{2C}\right) = l. \tag{S.22}$$

At this point, we need to solve the last equation numerically for given D and l (see the problem of soap films). From the last equation and (S.21) we have $\lambda = -\mathrm{sign}(C)\sqrt{1+(l/2)^2}$. The final solution up to the constant C is:

$$y(x) = C\cosh\left(\frac{x}{C}\right) - \mathrm{sign}(C)\sqrt{C^2+(l/2)^2}. \tag{S.23}$$

As long as $l > D$, eq. (S.22) has two solutions $C = \pm|C|$ and $C > 0$ is the one which gives the shape minimizing the potential energy ($C < 0$ gives the one which maximizes the potential energy) (Fig. 3).

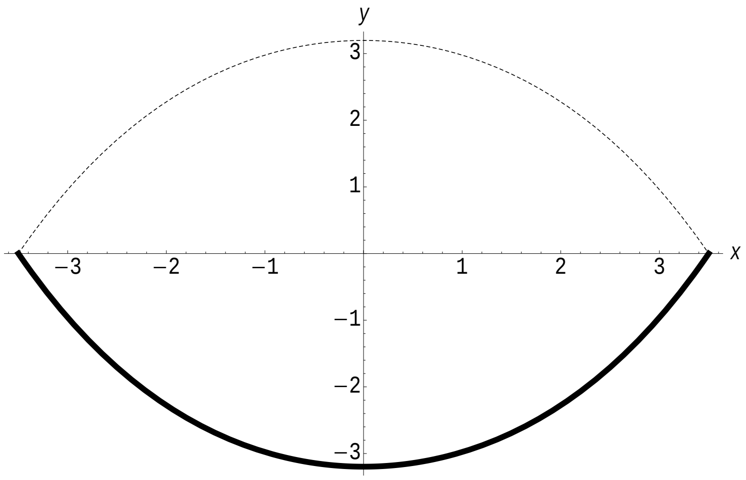


Figure 3: The shape of the bridge which minimizes (the thick curve) or maximizes (the dashed curve) the potential energy. For this plot $l = 10$ and $D = 7$ and eq. (S.22) has two solutions $C \approx \pm 2.311$.