Exercise 1. ***The Optimal Tunnel***

Suppose that we can build a tunnel through the Earth's crust connecting a city A to another city B (Fig. 1). If the friction is negligible, a train departing A with zero velocity would accelerate as the train gets closer to the center and decelerate as it moves away from the center. Due to energy conservation, the train would arrive at B with exactly zero velocity. In this problem we want to determine the profile of the tunnel that will be traversed in the shortest time.

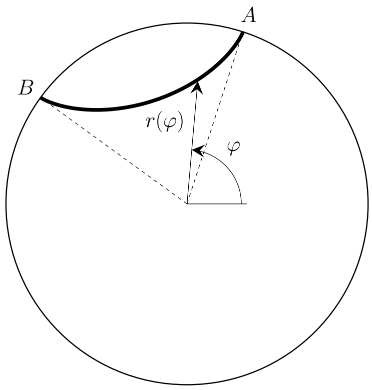

Figure 1: Sketch of an imaginary tunnel connecting cities A and B.

Model the Earth as a uniform solid sphere of radius R and mass M with constant density. Knowing that the tunnel lies in the plane defined by the two cities and the center of earth, parametrize it with a curve $(\varphi, r(\varphi))$. The goal is to find the curve that gives the shortest amount of time when the train moves only due to gravity.

Note that this problem is very similar to the brachistochrone discussed in the lecture notes. The main difference is that in this problem the direction and the strength of the gravitational force changes along the path. At a point $(\varphi, r(\varphi))$ with distance r from the center, the acceleration due to gravity points toward the center and has a magnitude of

$$|g_r| = \frac{GM}{R^3} r = g\frac{r}{R}, \tag{1}$$

where G is Newton's gravitational constant.

(a) Using the conservation of energy show that the velocity of the train at point $(\varphi, r(\varphi))$ is given by

$$v = \sqrt{\frac{g}{R}}\sqrt{R^2 - r^2}. \tag{2}$$

(b) Show that for a given curve the travel duration is

$$T_{A\to B} = \sqrt{\frac{R}{g}} \int_{\varphi_A}^{\varphi_B} \frac{\sqrt{r^2 + r'(\varphi)^2}}{\sqrt{R^2 - r^2}} \mathrm{d}\varphi\,, \tag{3}$$

where φ_A (φ_B) denotes starting (end) angle of the two cities with respect to the origin of earth and some reference point.

(c) By using the Euler-Lagrange equation show that the curve that minimizes the travel time $T_{A\to B}$ is a *hypocycloid*, whose parametric equation is

$$r'(\varphi)^2 = \frac{R^2}{r_0^2}\left[\frac{r^2(r^2-r_0^2)}{R^2-r^2}\right], \tag{4}$$

where r_0 denotes the minimal distance to the center of earth.

(d) Rewrite the the expression for $T_{A\to B}$ as an integral over r instead of φ and use Eq. (4) to find

$$T_{A\to B} = \pi\sqrt{\frac{R^2-r_0^2}{gR}} \tag{5}$$

(e) To find the constant r_0 as a function of the distance between the two cities $\varphi_B - \varphi_A$, we need to actually solve the equation of motion. This is not straight forward, for which reason we want to use a shortcut, taking advantage of the optimal path beeing a hypocycloid. One way to construct a hypocycloid is to take a circle with radius R and a second one with radius $\rho < R$. The curve traced out by a fixed point on the smaller circle, while rolling it within the bigger circle, is a hypocycloid. Use this knowledge, to express the constant r_0 as a function of the distance $\varphi_B - \varphi_A$ between the two cities, and show that

$$T(\Delta\varphi) = \sqrt{\frac{R}{g}}\sqrt{\Delta\varphi(2\pi-\Delta\varphi)}\,, \qquad \Delta\varphi = \varphi_B - \varphi_A\,. \tag{6}$$

Solution.

(a) The potential energy $U(r,\varphi)$ and the kinetic energy $T(r,\varphi)$ are given by

$$U(r,\varphi) = \frac{1}{2R}(r^2-R^2)mg\,, \tag{S.1}$$
$$T(\dot r,\dot\varphi) = \frac{1}{2}mv(r,\varphi)^2\,. \tag{S.2}$$

Conservation of energy together with the initial condition $v(r=R,\varphi)=0$ results in

$$v(r,\varphi) = \sqrt{\frac{g}{R}}\sqrt{R^2-r^2}\,. \tag{S.3}$$

(b) Repeated change of variables gives:

$$T = \int_0^T \mathrm{d}t = \int_0^L \frac{dt}{ds}\mathrm{d}s = \int_0^L \frac{1}{v}\mathrm{d}s = \int_{\varphi_A}^{\varphi_B}\frac{1}{v}\frac{\mathrm{d}s}{\mathrm{d}\varphi}\mathrm{d}\varphi = \int_{\varphi_A}^{\varphi_B}\frac{1}{v}\sqrt{r^2+r'(\varphi)^2}\mathrm{d}\varphi\,. \tag{S.4}$$

(c) We want to minimize the integral

$$J = \int_{\varphi_A}^{\varphi_B} F[r(\varphi), r'(\varphi);\varphi]\ \mathrm{d}\varphi\,, \tag{S.5}$$

with

$$F[r(\varphi), r'(\varphi);\varphi] = \frac{\sqrt{r^2+r'^2}}{\sqrt{R^2-r^2}}\,, \tag{S.6}$$

up to a constant. The Euler-Lagrange equation, using Eq. (9) from exercise sheet 2, reads,

$$c = F - r'\frac{\partial F}{\partial r'} = \frac{\sqrt{r^2+r'^2}}{\sqrt{R^2-r^2}} - \frac{r'^2}{\sqrt{r^2+r'^2}\sqrt{R^2-r^2}} = \frac{r^2}{\sqrt{r^2+r'^2}\sqrt{R^2-r^2}}\,. \tag{S.7}$$

The constant can be obtained from the condition $[r(\varphi^*), r'(\varphi^*)] = [r_0, 0]$:

$$c = \frac{r_0}{\sqrt{R^2-r_0^2}}\,. \tag{S.8}$$

Equation (S.7) is then equivalent to

$$r'^2 = \frac{R^2}{r_0^2}\left[\frac{r^2(r^2-r_0^2)}{R^2-r^2}\right]\,. \tag{S.9}$$

(d) Instead of choosing to φ as an integration variable, we could have taken r in the last step of Eq. (S.4):

$$T_{A\to B} = \int_0^T \mathrm{d}t = \int_0^L \frac{\mathrm{d}t}{\mathrm{d}s} ds = \int_0^L \frac{1}{v} \mathrm{d}s = \int_{\varphi_A}^{\varphi_B} \frac{1}{v} \frac{\mathrm{d}s}{d\varphi} \mathrm{d}\varphi = \int_{r_0}^{r_A} \frac{1}{v} \sqrt{1 + [r\varphi'(r)]^2} \mathrm{d}r\,. \tag{S.10}$$

Using the inverse function theorem,

$$\varphi'(r(\varphi)) = \frac{1}{r'(\varphi)} \qquad \text{for} \qquad \varphi_A < \varphi < \varphi^*\,, \quad r(\varphi^*) = r_0\,, \tag{S.11}$$

we find

$$\varphi'^2 = \frac{r_0^2}{R^2} \left[\frac{R^2 - r^2}{r^2(r^2 - r_0^2)} \right] \tag{S.12}$$

and with it

$$\begin{aligned} T_{A\to B} &= 2\int_{r_0}^{r_A} \frac{1}{v} \sqrt{1 + [r\varphi'(r)]^2} \mathrm{d}r = 2\sqrt{\frac{R}{g}} \int_{r_0}^{r_A} \frac{1}{\sqrt{R^2 - r^2}} \sqrt{1 + \frac{r_0^2}{R^2} \left[\frac{R^2 - r^2}{r^2 - r_0^2} \right]} \mathrm{d}r\,, \\ &= 2\sqrt{\frac{R^2 - r_0^2}{gR}} \int_{r_0}^{r_A} \frac{r}{\sqrt{R^2 - r^2}\sqrt{r^2 - r_0^2}} \mathrm{d}r = \sqrt{\frac{R^2 - r_0^2}{gR}} \int_{r_0{}^2}^{r_A{}^2} \frac{1}{\sqrt{(R^2 - x)(x - r_0^2)}} \mathrm{d}x\,, \\ &= \pi\sqrt{\frac{R^2 - r_0^2}{gR}}\,. \end{aligned} \tag{S.13}$$

(e) When rolling the smaller circle within the bigger circle, we get a hypocycloid connecting two points on the larger circle once the smaller circle completed a full rotation. The condition of rolling translates then into

$$\rho 2\pi = R(\varphi_B - \varphi_A) = R\Delta\varphi\,. \tag{S.14}$$

In such a situation we find r_0 directly from

$$R - 2\rho = r_0\,, \tag{S.15}$$

from which we conclude

$$r_0 = R\left(1 - \frac{\Delta\varphi}{\pi}\right) \tag{S.16}$$

and therefore

$$T_{A\to B} = \sqrt{\frac{R}{g}} \sqrt{\Delta\varphi(2\pi - \Delta\varphi)}\,. \tag{S.17}$$

So from the South pole to the North pole ($\Delta\varphi = \pi$) it takes only ≈ 42 minutes! From Zürich to New York ($\Delta\varphi \approx 1$) it would take only ≈ 30 minutes!

Exercise 2. *Atwood's Machine*

In this exercise you will consider the Atwood's machine and learn how to use the Lagrangian formalism for this case.

A simple Atwood's machine consists of two different masses, m_1 and m_2 connected by a rope of length l, as shown in the figure.

a) Using the x as the generalized coordinate, first write down the potential energy U, and the kinetic energy T for the system.

b) Now using

$$L = T - U \tag{7}$$

write down the Lagrangian and find the equations of motion, in terms of x.

c) Solve the equation of motion in terms of the acceleration $\ddot{x}$. Can you use this equation to determine g and how can you make it more accurate?

d) Now obtain the same result by this time using the Newton's second law for each of the masses. Note the differences between the Lagrangian method and the Newtonian method.

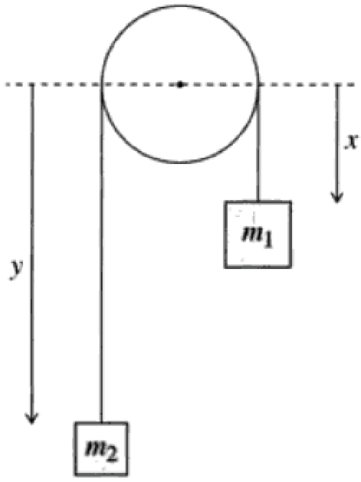

Figure 2: Atwood's Machine

Solution.

a) The potential energy is:

$$U = -m_1 g x - m_2 g (l - x)$$

while the kinetic energy is:

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

b) The Lagrangian has the form:

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1 g x + m_2 g (l - x)$$

Then we have:

$$\frac{\partial L}{\partial x} = (m_1 - m_2)\, g \tag{S.18}$$

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\, \dot{x} \tag{S.19}$$

Combining these we get:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$(m_1 - m_2)\, g - (m_1 + m_2)\, \ddot{x} = 0$$

c) Using the previous expression we have

$$\ddot{x} = \frac{(m_1 - m_2)}{(m_1 + m_2)}\, g$$

one can measure m_1, m_2 and a to determine g. If you choose m_1 and m_2 very close to one another then you can have the acceleration much smaller than g and have a more accurate determination of g in return.

d) Using the Newton's second law for each masses we write:

$$m_1 g - F_t = m_1 \ddot{x} \tag{S.20}$$

$$F_t - m_2 g = m_2 \ddot{x} \tag{S.21}$$

where F_t is the tension in the rope. Extracting an expression for F_t from the first equation and plugging it to the second we find the same result:

$$F_t = m_1\ddot{x} - m_1 g \tag{S.22}$$

$$(m_1 - m_2)\, g - (m_1 + m_2)\, \ddot{x} = 0 \tag{S.23}$$

In the Lagrangian approach one does not have to take into consideration any unknown force such as the tension of the rope.

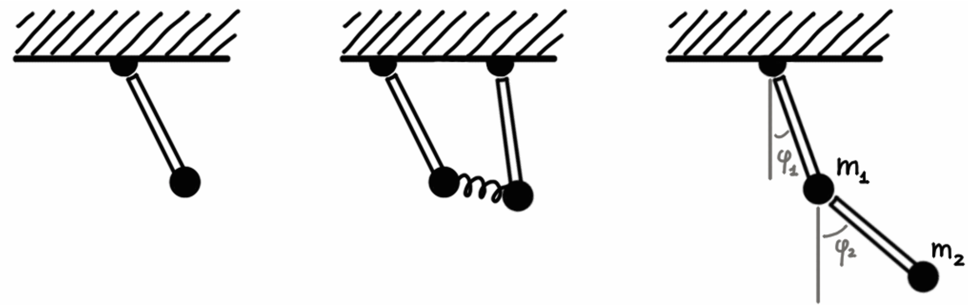

Figure 3: The three pendulum systems considered in exercise 3, 4 and 5.

Exercise 3. *Simple pendulum*

Consider a simple, ideal pendulum of length R that oscillates in the x-z plane under the action of the gravitational force.

1) After having chosen an appropriate generalized coordinate, write down the lagrangian for the simple pendulum (without assuming small oscillations).
2) Derive the equation of motion.
3) How does it simplify when small oscillations are assumed?
4) Under this approximation, find an explicit solution for the system's motion. If you need integration constants, comment on their physical meaning.

Solution.

1) In this solution the angle ϕ with respect to the vertical will be used. In general, if possible, it is convenient to choose a set of coordinates q_i along which the motion is free. Another suggestion is to set their zeroes at an equilibrium position, when the latter is easy enough to guess. For instance, the length $x = R\phi$ of the circle arc described by the pendulum starting from the vertical would have been equally good in this case. For the kinetic and potential energy we have

$$T = \frac{1}{2} mR^2 \dot{\phi}^2, \qquad U = mgz = mgR(1 - \cos\phi); \tag{S.24}$$

which gives up to an additive constant

$$L[\phi, \dot{\phi}] = T - U = \frac{1}{2} mR^2 \dot{\phi}^2 + mgR\cos\phi. \tag{S.25}$$

2) The Euler-Lagrange equation that minimizes the action reads

$$\frac{\partial L}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad -mgR\sin\phi - \frac{\mathrm{d}}{\mathrm{d}t} mR^2\dot{\phi} = 0 \quad \Rightarrow \quad g\sin\phi + R\ddot{\phi} = 0. \tag{S.26}$$

3) For $\phi \ll 1$ one has $\sin\phi = \phi + \mathcal{O}(\phi^3)$, thus getting

$$\ddot{\phi} + \omega^2\phi = 0, \tag{S.27}$$

where we defined $\omega^2 \equiv g/R$.

4) The differential equation above is that of a simple harmonic oscillator, which is solved e.g. by

$$\phi(t) = A\sin(\omega t) + B\cos(\omega t). \tag{S.28}$$

In order to interpret the integration constants A and B we observe

$$\phi(0) = B, \qquad \dot\phi(0) = \omega A, \tag{S.29}$$

which means that B is the starting position of the pendulum and ωA its initial velocity.

Exercise 4. *Coupled pendulum*

Now consider two identical pendulums with the same characteristics, attached to the same roof at a distance d from each other along the x axis. In addition, the two weights are coupled by an ideal spring of characteristic constant k and length at rest d.

1) Write down the Lagrangian for the described system.
2) Derive the equations of motion.
3) Simplify the system of differential equations in the case of small oscillations around the equilibrium position.
4) Diagonalize and solve the system of differential equations.
5) Comment on the meaning of the variables that allow equations of motion to be decoupled.

Solution. We shall use the two angles ϕ_1 and ϕ_2 of the pendulums as our coordinates. Then we find

$$T = \frac{l^2 m}{2}(\dot\phi_1^2 + \dot\phi_2^2),$$

$$U = \frac{k}{2}(\Delta x)^2 + mgl(1-\cos\phi_1) + mgl(1-\cos\phi_2)$$

with

$$(\Delta x)^2 = \left(\sqrt{(d + l\sin\phi_1 - l\sin\phi_2)^2 + (l\cos\phi_1 - l\cos\phi_2)^2} - d\right)^2,$$

and so we can write the Lagrangian as

$$L = T - U = \frac{l^2 m}{2}(\dot\phi_1^2 + \dot\phi_2^2) - \frac{k}{2}(\Delta x)^2 - mgl(1-\cos\phi_1) - mgl(1-\cos\phi_2).$$

The equations of motions are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi_1} = \frac{\partial L}{\partial\phi_1} \tag{S.30}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi_2} = \frac{\partial L}{\partial\phi_2} \tag{S.31}$$

(S.32)

which in this case evaluates to

$$m\ddot\phi_1 = -\frac{1}{2l^2}k\frac{\partial(\Delta x)^2}{\partial\phi_1} - \frac{mg}{l}\sin\phi_1$$

and

$$m\ddot\phi_2 = -\frac{1}{2l^2}k\frac{\partial(\Delta x)^2}{\partial\phi_2} - \frac{mg}{l}\sin\phi_2.$$

For the case of small oscillations, we have $\sin\phi \approx \phi$, $\cos\phi \approx 1$ for ϕ_1 and ϕ_2 so we can write

$$\begin{aligned}(\Delta x)^2 &\approx \left(\sqrt{(d + l\phi_1 - l\phi_2)^2} - d\right)^2 \\ &= l^2(\phi_1 - \phi_2)^2,\end{aligned}$$

so that $\frac{\partial(\Delta x)^2}{\partial\phi_1} \approx 2l^2(\phi_1 - \phi_2)$, and so to first order in the angles, the equations of motion simplify to

$$\begin{pmatrix} m\ddot{\phi}_1 \\ m\ddot{\phi}_2 \end{pmatrix} = \begin{pmatrix} -mg/l - k & k \\ k & -mg/l - k \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

In order to solve the system of equations, we note that one eigenvector is given by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the other by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The system of equations then simplifies to $(\ddot{\phi}_1+\ddot{\phi}_2) = -\frac{g}{l}(\phi_1+\phi_2)$ and $(\ddot{\phi}_1-\ddot{\phi}_2) = -(\frac{g}{l}+\frac{2k}{m})(\phi_1-\phi_2)$ respectively. This gives

$$(\phi_1 + \phi_2) = A\cos\left(\sqrt{\frac{g}{l}}t\right) + B\sin\left(\sqrt{\frac{g}{l}}t\right) \tag{S.33}$$

$$(\phi_1 - \phi_2) = C\cos\left(\sqrt{\frac{g}{l} + \frac{2k}{m}}t\right) + D\sin\left(\sqrt{\frac{g}{l} + \frac{2k}{m}}t\right), \tag{S.34}$$

with A, B, C, D constants determined by the initial conditions. The two decoupled equations above define the *normal modes* of the oscillation (the first one corresponds to the mode where the two pendulums swing in phase, the second one where they swing in opposite directions). Any general solution is a combination of the two modes.

Exercise 5. *Double pendulum*

Consider a pendulum attached to the end of another - a system also known as "double pendulum". Call the generic lengths of the pendulums R_1 and R_2, and take their masses to be m_1 and m_2 respectively.

1) Write down the lagrangian in terms of the angles ϕ_1 and ϕ_2 indicated in the picture.
   Hint. In order to make calculations more manageable, check if you can use $\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2 = \cos(\phi_1 - \phi_2)$.
2) Derive the equations of motion.
3) Simplify the system of differential equations in the case of small oscillations around the equilibrium position.
   Hint. To understand which terms are small, it is convenient to set $\phi_i(t) = \varepsilon\varphi_i(t)$ *and take the limit* $\varepsilon \to 0$ *assuming all other parameters are of order* 1.

Now consider the somewhat simpler case $m_1 = m_2 = m$, $R_1 = R_2 = R$. Assuming that the solutions can be written as a linear combination of normal modes, i.e. simple oscillatory solutions, corresponds to making the ansatz

$$\varphi_j(t) = \mathrm{Re}\sum_k A_{jk}e^{i\alpha_k t}. \tag{8}$$

4) Solve the system of differential equations to find the configuration of the double pendulum as a function of time.

Solution.

1) Given the angles in the picture and an x-z reference frame centered on the pivot of the upper pendulum, one has

$$\begin{cases} x_1 = R_1\sin\phi_1, \\ z_1 = -R_1\cos\phi_1, \end{cases} \qquad \begin{cases} x_2 = R_1\sin\phi_1 + R_2\sin\phi_2, \\ z_2 = -R_1\cos\phi_1 - R_2\cos\phi_2. \end{cases} \tag{S.35}$$

The kinetic energies then read

$$\begin{cases} T_1 = \dfrac{1}{2}m_1(\dot{x}_1^2 + \dot{z}_1^2) = \dfrac{1}{2}m_1 R_1^2\dot{\phi}_1^2, \\ T_2 = \dfrac{1}{2}m_2(\dot{x}_2^2 + \dot{z}_2^2) = \dfrac{1}{2}m_2[R_1^2\dot{\phi}_1^2 + R_2^2\dot{\phi}_2^2 + 2R_1R_2\cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2], \end{cases} \tag{S.36}$$

and the potentials are

$$\begin{cases} U_1 = m_1 g z_1 = -m_1 g R_1 \cos\phi_1, \\ U_2 = m_2 g z_2 = -m_2 g(R_1\cos\phi_1 + R_2\cos\phi_2). \end{cases} \tag{S.37}$$

Therefore the lagrangian is simply

$$L[\phi_1,\phi_2,\dot{\phi}_1,\dot{\phi}_2] = T_1 + T_2 - U_1 - U_2. \tag{S.38}$$

2) The Euler-Lagrange equations read

$$\begin{cases} \dfrac{\partial L}{\partial \phi_1} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{\phi}_1} = 0, \\ \dfrac{\partial L}{\partial \phi_2} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{\phi}_2} = 0; \end{cases} \tag{S.39}$$

and they expand to

$$\begin{cases} \dfrac{\partial T_2}{\partial \phi_1} - \dfrac{\partial U_1}{\partial \phi_1} - \dfrac{\partial U_2}{\partial \phi_1} = \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial T_1}{\partial \dot{\phi}_1} + \dfrac{\partial T_2}{\partial \dot{\phi}_1}\right], \\ \dfrac{\partial T_2}{\partial \phi_2} - \dfrac{\partial U_2}{\partial \phi_2} = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial T_2}{\partial \dot{\phi}_2}; \end{cases} \tag{S.40}$$

where terms that are obviously zero have not been written down. Computing derivatives we find

$$\begin{cases} -m_2R_1R_2\sin(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 - (m_1+m_2)gR_1\sin\phi_1 = \dfrac{\mathrm{d}}{\mathrm{d}t}\left[(m_1+m_2)R_1^2\dot{\phi}_1 + m_2R_1R_2\cos(\phi_1-\phi_2)\dot{\phi}_2\right], \\ m_2R_1R_2\sin(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2 - m_2gR_2\sin\phi_2 = \dfrac{\mathrm{d}}{\mathrm{d}t}\left[m_2R_2^2\dot{\phi}_2 + m_2R_1R_2\cos(\phi_1-\phi_2)\dot{\phi}_1\right]; \end{cases}$$

$$\begin{cases} -(m_1+m_2)g\sin\phi_1 = (m_1+m_2)R_1\ddot{\phi}_1 + m_2R_2[\cos(\phi_1-\phi_2)\ddot{\phi}_2 + \sin(\phi_1-\phi_2)\dot{\phi}_2^2], \\ -m_2g\sin\phi_2 = m_2R_2\ddot{\phi}_2 + m_2R_1[\cos(\phi_1-\phi_2)\ddot{\phi}_1 - \sin(\phi_1-\phi_2)\dot{\phi}_1^2]. \end{cases} \tag{S.41}$$

3) Setting $\phi_i = \varepsilon\varphi_i$ and keeping all terms of order ε we find

$$\begin{cases} -(m_1+m_2)g\varphi_1 = (m_1+m_2)R_1\ddot{\varphi}_1 + m_2R_2\ddot{\varphi}_2, \\ -m_2g\varphi_2 = m_2R_2\ddot{\varphi}_2 + m_2R_1\ddot{\varphi}_1. \end{cases} \tag{S.42}$$

4) For equal masses and lengths one has

$$\begin{cases} 2\ddot{\varphi}_1 + \ddot{\varphi}_2 + 2\omega^2\varphi_1 = 0, \\ \ddot{\varphi}_1 + \ddot{\varphi}_2 + \omega^2\varphi_2 = 0; \end{cases} \tag{S.43}$$

where $\omega^2 = g/R$. Substituting the normal modes decomposition into the equations of motion gives

$$\begin{cases} \mathrm{Re}\sum_k \left[(-2A_{1k}\alpha_k^2 - A_{2k}\alpha_k^2 + 2\omega^2A_{1k})e^{i\alpha_k t}\right] = 0, \\ \mathrm{Re}\sum_k \left[(-A_{1k}\alpha_k^2 - A_{2k}\alpha_k^2 + \omega^2A_{2k})e^{i\alpha_k t}\right] = 0. \end{cases} \tag{S.44}$$

Since we would like $\phi_j(t)$ to be a solution for every time t, it is necessary that

$$\begin{cases} -2A_{1k}\alpha_k^2 - A_{2k}\alpha_k^2 + 2\omega^2A_{1k} = 0, \\ -A_{1k}\alpha_k^2 - A_{2k}\alpha_k^2 + \omega^2A_{2k} = 0; \end{cases} \tag{S.45}$$

for every normal mode k separately (no sum over k!). The system of equations for the coefficients A_{1k} and A_{2k} reads

$$\begin{cases} 2(\omega^2-\alpha_k^2)A_{1k} - \alpha_k^2A_{2k} = 0, \\ -\alpha_k^2A_{1k} + (\omega^2-\alpha_k^2)A_{2k} = 0; \end{cases} \tag{S.46}$$

and, being a homogeneous linear system, it has nontrivial solutions if and only if the determinant of the coefficient matrix vanishes:

$$\begin{aligned} 2(\omega^2-\alpha_k^2)^2-\alpha_k^4&=0,\\ \alpha_k^4-4\omega^2\alpha_k^2+2\omega^4&=0. \end{aligned} \tag{S.47}$$

This quartic equation admits the solutions

$$\alpha_k^2=(2\pm\sqrt{2})\omega^2 \qquad \Rightarrow \qquad \alpha_{1,3}=\pm\sqrt{2+\sqrt{2}}\,\omega, \qquad \alpha_{2,4}=\pm\sqrt{2-\sqrt{2}}\,\omega. \tag{S.48}$$

The coefficients are then constrained to

$$(2+\sqrt{2})A_{1k}+(1+\sqrt{2})A_{2k}=0 \qquad \text{if} \quad k=1,3; \tag{S.49}$$

$$(2-\sqrt{2})A_{1k}-(\sqrt{2}-1)A_{2k}=0 \qquad \text{if} \quad k=2,4. \tag{S.50}$$

This yields the solutions

$$\varphi_1(t)=+c_1(1+\sqrt{2})\cos[\alpha_1(t+t_1)]+c_2(\sqrt{2}-1)\cos[\alpha_2(t+t_2)], \tag{S.51}$$

$$\varphi_2(t)=-c_1(2+\sqrt{2})\cos[\alpha_1(t+t_1)]+c_2(2-\sqrt{2})\cos[\alpha_2(t+t_2)]. \tag{S.52}$$