Exercise 1. ***The Galilean and the Euclidean group***

In this exercise we are going to review the basic properties of some of the groups that appear in classical mechanics and that describe the invariance of physical laws with respect to transformation between coordinate systems. As seen in the lecture, for non-relativistic velocities $v \ll c$, different coordinate systems S and S' describing the same Euclidean spacetime $\mathbb{E} = \mathbb{R} \oplus \mathbb{R}^3$ are related by a Galilean transformation

$$\mathcal{G} : \mathbb{E} \to \mathbb{E}, \quad (t, \vec{r}) \mapsto (t', \vec{r}') = \mathcal{G}[(t, \vec{r})] \equiv (t + t_0, \boldsymbol{R}\vec{r} - \vec{r}_0 - \vec{V}t), \tag{1}$$

where t_0 is the time shown by the clock of S' at time $t = 0$ in S, $\vec{r}_0$ is the translation of the origin of S' with respect to the origin of S, and $\vec{V}$ is the apparent velocity of a point in S' according to an observer in S, and $\boldsymbol{R}$ is a rotation of the axes of S' with respect to the axes of S. Recall the definition of a group:

Definition 1. Given a set G and a binary operation $\circ$: $G \times G \to G$, the algebraic structure $(G, \circ)$ is called a group if it satisfies the following requirements (group axioms):

1. Closure: $\forall a, b \in G \Rightarrow a \circ b \in G$
2. Associativity: $\forall a, b, c \in G \Rightarrow (a \circ b) \circ c = a \circ (b \circ c)$
3. Identity element: $\exists e \in G : \ \forall a \in G : e \circ a = a \circ e = a$
4. Inverse element: $\forall a \in G \ \exists a^{-1} \in G : a^{-1} \circ a = a \circ a^{-1} = e$

(a) i) Show that the set of all Galilean transformations $\mathcal{G}$, together with composition of transformations, forms a group SGal(3), called *(special) Galilean group*. Describe the general features of this group. How many parameters are necessary to completely describe it?

ii) Show that the three-dimensional rotations $\boldsymbol{R}$ form a proper subgroup SO(3) $<$ SGal(3).

Hint. To show that the set H is a subgroup of G, $H < G$, it is not necessary to check again all the group axioms for H. Instead it is sufficient to show that, for two elements h_1, $h_2 \in H$, the product $h_1 \circ h_2^{-1}$ always lies within the subspace. This is often called the "subgroup test".

iii) Show that the composition of spatial translations $\mathbb{R}^3$ and proper rotations SO(3) forms a subgroup SEucl(3) $<$ SGal(3), called *(special) Euclidean group*. Can any element of SEucl(3) be written as a simple product of an element of $\mathbb{R}^3$ with and element of SO(3)?

Hint. An element $\mathcal{S} \in$ SEucl(3) acts on $\mathbb{R}^3$ as follows:

$$\mathcal{S} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \vec{r} \mapsto \mathcal{S}(\vec{r}) \equiv \boldsymbol{R}\vec{r} - \vec{r}_0 \tag{2}$$

(b) The (Newtonian) principle of relativity states that the laws of classical mechanics are invariant under Galilean transformations. Does this principle extend to the laws of electrodynamics? Consider the Lorentz force

$$\vec{F}_L = q(\vec{E} + \vec{V} \times \vec{B}), \tag{3}$$

where q is the charge, $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field.

i) How should the fields transform under a boost with velocity $\vec{V}_0$ to assure Galilean invariance of the Lorentz force for every reference frame?

ii) Now consider Gauss' law in vacuum:

$$\vec{\nabla} \cdot \vec{E} = 0. \tag{4}$$

Is this equation Galilean invariant?

Solution.

(a) i) In the following we use tensor notation and Einstein's summation convention. Let $\mathcal{G}_i$ indicate a Galilean transformation with parameters t_i (time translation), $\vec{r}_i$ (origin shift), R_i (axis rotation) and $\vec{V}_i$ (velocity boost).

Closure: The composition of two Galilean transformations $\mathcal{G}_0$ and $\mathcal{G}_1$ is:

$$(\mathcal{G}_1 \circ \mathcal{G}_0)[(t, r_i)] = \mathcal{G}_1[(t + t_0, R_{0,ij}r_j - r_{0,i} - V_{0,i}t)] \tag{S.1}$$
$$= (t + t_1 + t_0, R_{1,il}R_{0,lj}r_j - R_{1,il}r_{0,l} - r_{1,i} - (R_{1,il}V_{0,l} + V_{1,i})t) \tag{S.2}$$
$$= (t + \tilde{t}, \tilde{R}_{ij}r_j - \tilde{r}_i - \tilde{V}_i t) \tag{S.3}$$

where we have implicitly defined the new group element $\tilde{\mathcal{G}}$, which is a well-defined Galilean transformation with parameters:

$$\tilde{t} = t_1 + t_0 \tag{S.4}$$
$$\tilde{R}_{ij} = R_{1,il}R_{0,lj} \tag{S.5}$$
$$\tilde{r}_i = R_{1,il}r_{0,l} + r_{1,i} \tag{S.6}$$
$$\tilde{V}_i = R_{1,il}V_{0,l} + V_{1,i}. \tag{S.7}$$

Associativity: Let $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$ be three different Galilean transformations. Then the composition of the three is:

$$((\mathcal{G}_2 \circ \mathcal{G}_1) \circ \mathcal{G}_0)[(t, r_i)] = (\mathcal{G}_2 \circ \mathcal{G}_1)[(\underbrace{t + t_0}_{\equiv t'}, \underbrace{R_{0,ij}r_j - r_{0,i} - V_{0,i}t}_{\equiv r'_i})] \tag{S.8}$$
$$= (\mathcal{G}_2 \circ \mathcal{G}_1)[(t', r'_i)] \tag{S.9}$$
$$= (t' + \tilde{\tilde{t}}, \tilde{\tilde{R}}_{ij}r'_j - \tilde{\tilde{r}}_i - \tilde{\tilde{V}}_i t') \tag{S.10}$$
$$= (t + t_2 + t_1 + t_0, R_{2,ik}R_{1,kl}R_{0,lj}r_j - R_{2,ik}(R_{1,kl}r_{0,l} + r_{1,k}) - r_{2,i} + \tag{S.11}$$
$$- (R_{2,ik}(R_{1,kl}V_{0,l} + V_{1,k}) + V_{2,i})t) \tag{S.12}$$
$$= (\tilde{t} + t_2, R_{2,ik}\tilde{R}_{kj}r_j - R_{2,ik}\tilde{r}_k - r_{2,i} - (R_{2,ik}\tilde{V}_k + V_{2,i})t) \tag{S.13}$$
$$= (\mathcal{G}_2 \circ (\mathcal{G}_1 \circ \mathcal{G}_0))[(t, r_i)] \tag{S.14}$$

where the double-tilde parameters

$$\tilde{\tilde{t}} = t_2 + t_1 \tag{S.15}$$
$$\tilde{\tilde{R}}_{ij} = R_{2,il}R_{1,lj} \tag{S.16}$$
$$\tilde{\tilde{r}}_i = R_{2,il}r_{1,l} + r_{2,i} \tag{S.17}$$
$$\tilde{\tilde{V}}_i = R_{2,il}V_{1,l} + V_{2,i}. \tag{S.18}$$

have been defined for $\mathcal{G}_1$ and $\mathcal{G}_2$ in the same way as the single-tilde for $\mathcal{G}_0$ and $\mathcal{G}_1$ have been.

Identity element: Let $\mathcal{E}$ be the Galilean transformations with parameters $t_0 = 0$, $\vec{r}_0 = \vec{V} = \vec{0}$ and $\boldsymbol{R} = \mathbb{1}$. Then:

$$\mathcal{E}[(t, r_i)] = (t + t_0, R_{0,ij}r_j - r_{0,i} - V_i t) = (t + 0, \delta_{ij}r_j - \vec{0}) = (t, r_i). \tag{S.19}$$

Hence $\mathcal{E}$ is the identity element of the Galilean group.

Inverse element: Given an arbitrary Galilean transformation $\mathcal{G}$ with parameters t_0, $\vec{r}_0$, $\vec{V}$ and $\boldsymbol{R}$, let $\mathcal{H}$ be the Galilean transformation with parameters $t_H = -t_0$, $\vec{r}_H = -\boldsymbol{R}^{-1}\vec{r}_0$, $\vec{V}_H = \boldsymbol{R}^{-1}\vec{V}$ and $\boldsymbol{R}_H = \boldsymbol{R}^{-1}$ (which is well-defined because $\boldsymbol{R}\boldsymbol{R}^T = \mathbb{1}$). Then a composition of $\mathcal{G}$ with $\mathcal{H}$ gives

$$\begin{aligned}\mathcal{G}\circ\mathcal{H}[(t,r_i)] &= \mathcal{G}[(t-t_0, R^{-1}_{ij}r_j + R^{-1}_{ij}r_{0,j} + R^{-1}_{ij}V_j t)] = && \text{(S.20)}\\ &= (t+t_0-t_0, R_{ik}R^{-1}_{kj}r_j + R_{ik}R^{-1}_{kj}r_{0,j} + R_{ik}R^{-1}_{kj}V_j t - r_{0,i} - V_i t) && \text{(S.21)}\\ &= (t, \delta_{ij}r_j + \delta_{ij}r_{0,j} - r_{0,i} + \delta_{ij}V_j t - V_i t) && \text{(S.22)}\\ &= (t, r_i) && \text{(S.23)}\\ &= \mathcal{E}(t, r_i). && \text{(S.24)}\end{aligned}$$

and likewise if $\mathcal{G}$ and $\mathcal{H}$ are switched. Thus, $\mathcal{H} = \mathcal{G}^{-1}$ is the inverse of the Galilean transformation $\mathcal{G}$. This last property concludes the proof that SGal(3) is a group.

There are three spatial dimensions and both shifts and boosts can be regarded as automorphisms of $\mathbb{R}^3$. To fully describe a three-dimensional rotation $\boldsymbol{R}$, three angular parameters are required. Together with the one-dimensional time translation, the total number of parameters required to fully describe a Galilean transformation is $3 + 3 + 3 + 1 = 10$. The group is continuous, since the transformations are, hence it is a Lie group. It is not an abelian group, since two Galilean transformations do not necessarily commute with each other. In fact, while translations and boosts are commutative, rotations are not, which can be seen by looking at their matrix representation: $\boldsymbol{R}_a\boldsymbol{R}_b \neq \boldsymbol{R}_b\boldsymbol{R}_a$.

ii) Let $\boldsymbol{R}_a$, $\boldsymbol{R}_b \in \mathrm{SO}(3)$ be two arbitrary rotations between coordinate systems. As mentioned in the hint, to show that they span a subgroup, we do not have to prove again all the axioms, but it is sufficient to show that $\boldsymbol{R}_c = \boldsymbol{R}_a\boldsymbol{R}_b^{-1}$ lies within the subspace. This is often called the "subgroup test". In fact, if the subgroup is non-empty, associativity in the subgroup follows from the associativity of the bigger group, and both the identity element $\mathcal{E} = \mathcal{G}\circ\mathcal{G}^{-1}$ and the inverse element $\mathcal{G}^{-1} = \mathcal{E}\circ\mathcal{G}^{-1}$ can be written in the form above for some element $\mathcal{G}$. Furthermore, $\mathcal{H}\circ\mathcal{G} = \mathcal{H}\circ(\mathcal{G}^{-1})^{-1}$ and the subgroup is closed under the operation $\circ$. From the properties of rotation matrices, namely

$$\boldsymbol{R}\boldsymbol{R}^T = \mathbb{1} \quad \text{and} \quad \det\boldsymbol{R} = 1, \tag{S.25}$$

follows:

$$\begin{aligned}\boldsymbol{R}_c\boldsymbol{R}_c^T &= (\boldsymbol{R}_a\boldsymbol{R}_b^{-1})(\boldsymbol{R}_a\boldsymbol{R}_b^{-1})^T && \text{(S.26)}\\ &= \boldsymbol{R}_a\boldsymbol{R}_b^{-1}(\boldsymbol{R}_b^{-1})^T\boldsymbol{R}_a^T && \text{(S.27)}\\ &= \boldsymbol{R}_a\boldsymbol{R}_b^{-1}\boldsymbol{R}_b\boldsymbol{R}_a^{-1} && \text{(S.28)}\\ &= \boldsymbol{R}_a\mathbb{1}\boldsymbol{R}_a^{-1} && \text{(S.29)}\\ &= \mathbb{1} && \text{(S.30)}\end{aligned}$$

and

$$\begin{aligned}\det[\boldsymbol{R}_c] &= \det[\boldsymbol{R}_a\boldsymbol{R}_b^{-1}] && \text{(S.31)}\\ &= \det[\boldsymbol{R}_a]\det[\boldsymbol{R}_b^{-1}] && \text{(S.32)}\\ &= \det[\boldsymbol{R}_a]\frac{1}{\det[\boldsymbol{R}_b]} && \text{(S.33)}\\ &= 1. && \text{(S.34)}\end{aligned}$$

Hence, SO(3) is a proper subgroup of SGal(3).

iii) First note that from part i) the inverse element of a Euclidean transformation involves composition of the inverses of $\mathbb{R}^3$ ($-\vec{r}_0$) and SO(3) ($\boldsymbol{R}^T$):

$$\mathcal{S}^{-1}(\vec{r}) = \boldsymbol{R}^T\vec{r} + \boldsymbol{R}^T\vec{r}_0 \tag{S.35}$$

so SEucl(3) can not be a direct product of the two subgroups and there are no $\vec{v}\in\mathbb{R}^3$ and $\boldsymbol{R}\in\mathrm{SO}(3)$ s.t. $\mathcal{S} = \boldsymbol{R}\otimes\vec{v}$. Now let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two Euclidean transformations. Then again it is sufficient to probe that $\mathcal{S}_2 \circ \mathcal{S}_1^{-1}$ is also Euclidean:

$$\mathcal{S}_2 \circ \mathcal{S}_1^{-1}(r_i) = \mathcal{S}_2(R_{1,ij}^{-1} r_j + R_{1,ij}^{-1} r_{1,j}) \tag{S.36}$$
$$= \underbrace{R_{2,ik} R_{1,kj}^{-1}}_{\equiv R_{3,ij}} r_j + \underbrace{R_{2,ik} R_{1,kj}^{-1} r_{1,j} - r_{2,i}}_{\equiv r_{3,i}} \tag{S.37}$$
$$= R_{3,ij} r_j - r_{3,i}. \tag{S.38}$$

The Euclidean group is six-dimensional, being parametrized by three angles and three translations, and describes all transformations which leave the Euclidean geometry invariant.

(b) i) Galilean invariance of the equation of motion $\vec{F}_L = m\vec{a}$ implies Galilean invariance of the Lorentz force itself, since the acceleration (for velocities $\ll c$) is Galilean invariant as stated in the lecture. Hence, by applying a Galilean boost $\vec{V}_0$ to the force one obtains (note that the charge q is Galilean invariant as it does not depend on the reference frame):

$$\vec{F} = \vec{F}' \tag{S.39}$$
$$\vec{E} + \vec{V} \times \vec{B} = \vec{E}' + \vec{V}' \times \vec{B}' \tag{S.40}$$
$$\vec{E} + \vec{V} \times (\vec{B} - \vec{B}') = \vec{E}' - \vec{V}_0 \times \vec{B}' \tag{S.41}$$

since $\vec{V}' = \vec{V} - \vec{V}_0$ because $\vec{V}' = \frac{\partial \vec{r}'}{\partial t'} = \frac{\partial \vec{r}'}{\partial t}$ and $\vec{r}' = \vec{r} - V_0 t$. For the above equality to hold independently of any choice of $\vec{V}$, the fields must transform according to:

$$\vec{B}'(\vec{r}', t') = \vec{B}(\vec{r}, t) \tag{S.42}$$
$$\vec{E}'(\vec{r}', t') = \vec{E}(\vec{r}, t) + \vec{V}_0 \times \vec{B}(\vec{r}, t). \tag{S.43}$$

Note that even in the absence of an electric field in the original reference frame S, an observer in the reference frame S' measures an electric field $\vec{E}' = \vec{V}_0 \times \vec{B}$! This peculiarity suggests that electricity and magnetism are actually two different manifestations of the same physical phenomenon. The transformations above are the so-called magnetoquasistatic approximation (or limit) of the more general Lorentz transformations, which hold also when special relativity is considered.

ii) Recall from the lecture that $\vec{\nabla}$ is a vector and therefore it is invariant under boosts. Using the transformation for the electric field previously obtained and the identity

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}), \tag{S.44}$$

from exercise 2 of series 1, we can calculate the transformation of Gauss' law in the new reference frame:

$$\vec{\nabla}' \cdot \vec{E}' = \vec{\nabla} \cdot (\vec{E} + \vec{V}_0 \times \vec{B}) \tag{S.45}$$
$$= \vec{\nabla} \cdot \vec{E} + \vec{B} \cdot (\vec{\nabla} \times \vec{V}_0) - \vec{V}_0 \cdot (\vec{\nabla} \times \vec{B}) \tag{S.46}$$
$$= -\vec{V}_0 \cdot (\vec{\nabla} \times \vec{B}) \neq 0 \tag{S.47}$$

where in the last line we used Gauss' law in the original reference frame and the fact that $\vec{V}_0$ is a constant. Since the curl of the magnetic field is in general not zero, Gauss' law is not Galilean invariant.

Exercise 2. *Soap film*

Consider a soap film suspended between two circular wires of radius a parallel to the yz plane and centred at $x = \pm\ell$. The film will adjust its shape such that the surface energy is minimised.

You should assume that the film is very thin, its shape is cylindrically symmetric and that the surface energy is given by $E = \sigma S$ where σ is the surface tension and S is the area of the film. Gravity should be ignored throughout this exercise. This means that the film eventually reaches a shape with minimal surface area.

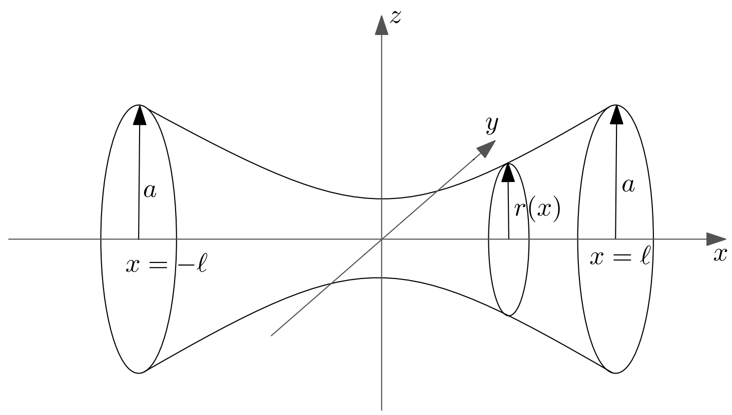

Figure 1: Soap film suspended between two circles

(a) Explain why the surface area can be written as

$$S[r] = \int_{-\ell}^{\ell} 2\pi r(x)\sqrt{1 + r'(x)^2}\, \mathrm{d}x \tag{5}$$

(b) Write down the Euler-Lagrange equations for minimising the functional $S[r]$ and show that they can be simplified to

$$\frac{r''r}{(1 + r'^2)^{3/2}} - \frac{1}{(1 + r'^2)^{1/2}} = 0. \tag{6}$$

Deduce that

$$\frac{r}{(1 + r'^2)^{1/2}} = c \tag{7}$$

for some constant c.

Note that in the steps above you first obtained a second-order differential equation and then transformed it into a much simpler first-order equation. There is actually a trick (known as the Beltrami identity) which allows you to write down immediately the first-order equation. In general, the functional to minimize/maximize is expressed as

$$T[y] = \int \mathrm{d}x\ F(y, y', x). \tag{8}$$

If $F(y, y', x) = F(y, y')$, i.e., the integrand in the functional does not depend explicitly on x, making use of Euler-Lagrange equations we will show that the quantity

$$H = F - y'\frac{\partial F}{\partial y'} \tag{9}$$

remains constant. This is proved by taking the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}H = \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y'' - y''\frac{\partial F}{\partial y'} - y'\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) \tag{10}$$

(remember that $\frac{\partial F}{\partial x} = 0$, so this term is not included above). Now we use the Euler-Lagrange equations $\frac{\partial F}{\partial y} = \frac{d}{dx}\frac{\partial F}{\partial y'}$ and see that all terms above cancel. Hence

$$F - y'\frac{\partial F}{\partial y'} = c\,, \tag{11}$$

where c is a constant. When we discuss Hamiltonian mechanics later in the class, we will see that the same result applies to an important physical quantity (the energy) when the variable x is the time.

(c) Let us now go back to the soap film problem. Derive the equation (7) again, this time using the Beltrami identity. You should be able to obtain a solution to this equation of the form $r = c\cosh((x - x_0)/c)$. Using the boundary conditions show that $x_0 = 0$ and explain graphically why there is no solution if the ratio a/ℓ is smaller than a certain value. What happens physically in such case?

Solution.

(a) The area of the film between x and $x+\mathrm{d}x$ is $\mathrm{d}A = 2\pi r(x)\sqrt{\mathrm{d}x^2 + \mathrm{d}r^2} = 2\pi r(x)\sqrt{1 + r'(x)^2}\mathrm{d}x$. Integrating yields the formula.

(b) Cancelling the constant pre-factor 2π the Euler-Lagrange equations become

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial}{\partial r'}\left(r\sqrt{1+r'^2}\right)\right] = \frac{\partial}{\partial r}\left[r\sqrt{1+r'^2}\right]$$

Now we expand the derivatives and make some algebra

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{rr'}{(1+r'^2)^{1/2}}\right] = (1+r'^2)^{1/2} \Rightarrow -\frac{rr'^2r''}{(1+r'^2)^{3/2}} + \frac{rr''+r'^2}{(1+r'^2)^{1/2}} = (1+r'^2)^{1/2} \Rightarrow$$
$$\Rightarrow \frac{1}{(1+r'^2)^{3/2}}\left[-rr'^2r'' + (1+r'^2)(rr''+r'^2) - (1+r'^2)^2\right] = 0 \Rightarrow \frac{1}{(1+r'^2)^{3/2}}\left[rr'' - 1 - r'^2\right] = 0$$
$$\Rightarrow \frac{rr''}{(1+r'^2)^{3/2}} - \frac{1}{(1+r'^2)^{1/2}} = 0$$

Note that the last equation is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{r}{(1+r'^2)^{1/2}}\right) = 0 \Rightarrow \frac{r}{(1+r'^2)^{1/2}} = \text{const.}$$

(c) Making use of the relation (11) we obtain

$$r\sqrt{1+r'^2} - \frac{r}{\sqrt{1+r'^2}}r'^2 = \frac{r}{\sqrt{1+r'^2}} = c.$$

as deduced previously. By squaring and solving for r' we obtain

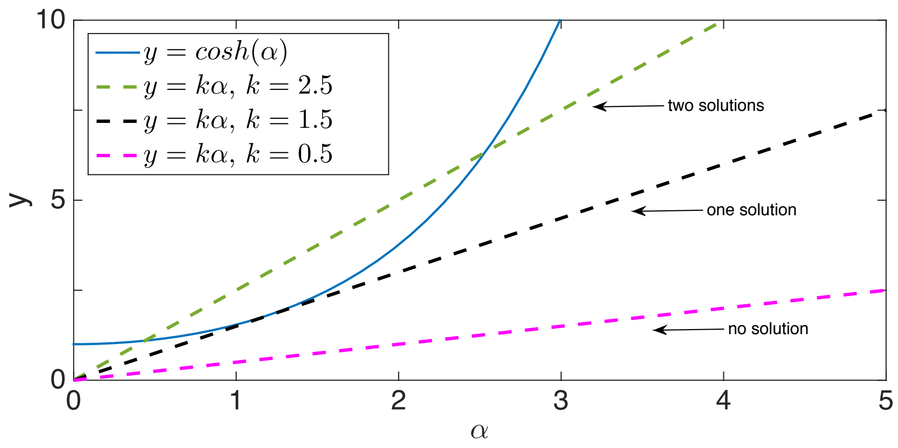

Figure 2: Graphic solution of the catenary boundary condition.

$$c^2(1+r'^2) = r^2 \quad \rightarrow \quad r' = \frac{dr}{dx} = \frac{1}{c}\sqrt{r^2-c^2},$$

where we took the positive square root. By separating the variables and integrating each side we obtain

$$\int_{r_0}^{r} \frac{\mathrm{d}z}{\sqrt{z^2-c^2}} = \int_{r_0}^{r} \frac{\mathrm{d}z/c}{\sqrt{(z/c)^2-1}} = \int_{\eta_0}^{\mathrm{acosh}(r/c)} \frac{\mathrm{d}\eta \sinh\eta}{\sqrt{\cosh^2\eta - 1}} = \mathrm{acosh}(r/c) + b = \frac{1}{c}\int_{x_0}^{x} \mathrm{d}z = (x-x_0)/c\,,$$

$$\text{i.e.,} \quad (x-x_0)/c = \mathrm{acosh}(r/c), \tag{S.48}$$

(Note that we could absorb the integration constant b into another constant x_0). This solution is called *catenary.* Applying the boundary conditions $r(\pm\ell) = a$ then implies $x_0 = 0$ and $a = c\cosh(\ell/c)$. Defining $\alpha = \ell/c$ changes this condition into $\frac{a}{\ell}\alpha = \cosh\alpha$. If the slope $k = a/\ell$ is not sufficiently large, this equation does not have a solution (see Fig. 2). In such case there is no smooth function $r(x)$ which minimises the functional $E[r]$ and satisfies the boundary conditions. Physically, the radius of the soap film will shrink and eventually the film will split into two parts, each of them filling one of the circular wires. The function describing such solution is not continuous, not twice differentiable and thus not obtainable in our calculation which assumes the existence and continuity of r and r'.

Exercise 3. *Geodesics on a Sphere*

In this problem, we want to show that the path of shortest distance between two points on the surface of a sphere lies along the great circle that connects the two points. In general, curves of minimum path length between two points are called *geodesics*. They are generalizations of the definition of a straight line in a curved space.

(a) Using spherical coordinates, show that the path length from point A to point B is given by the following integral

$$l_{A\to B} = \int_A^B \mathrm{d}l = \int_A^B \sqrt{(\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\phi)^2} \tag{12}$$

Since we want to find the path of least length on the surface of a sphere, we have the constraint $r = R$, where R is the radius of the sphere.

(b) Using the constraint $r = R$ write the path length (12) in the following form

$$l_{A\to B} = R\int_{\phi_A}^{\phi_B} F[\theta(\phi), \theta'(\phi)]\mathrm{d}\phi \tag{13}$$

Note that we chose to parametrize the path as $\theta(\phi)$ instead of the equivalent $\phi(\theta)$. Why?

(c) Show that the curve which minimizes the path length is given by

$$\phi(\theta) = \arcsin(C_1\cot\theta) + C_2, \tag{14}$$

where C_1 and C_2 are integration constant, which can be fixed by the requirement that the path passes through the points A and B.

Hint. Exploit the fact that the integrand $F[\theta(\phi), \theta'(\phi)]$ is independent of ϕ.

(d) Show that the solution (14) describes the great circle between points A and B.

Hint. Use the fact that a great circle is the intersection of a plane going through the origin of the coordinate system with the surface of the sphere.

Solution.

(a) The transformation from spherical coordinates to Cartesian coordinates reads

$$x = r\sin\theta\cos\phi, \tag{S.49}$$
$$y = r\sin\theta\sin\phi, \tag{S.50}$$
$$z = r\cos\theta. \tag{S.51}$$

From the above transformation it is straightforward to calculate the length element in spherical coordinates:

$$(\mathrm{d}l)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 = ... = (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 + r^2\sin^2\theta(\mathrm{d}\phi)^2. \tag{S.52}$$

Hence, the path length from point A to point B is given by the integral (12).

(b) The constraint implies that along the required path $r = R$ and $dr = 0$. Therefore the path length becomes

$$l_{A\to B} = \int_{\phi_A}^{\phi_B}\sqrt{R^2(\mathrm{d}\theta)^2 + R^2\sin^2\theta(\mathrm{d}\phi)^2} = R\int_{\phi_A}^{\phi_B}\sqrt{\theta'^2 + \sin^2\theta}\,\mathrm{d}\phi, \tag{S.53}$$

where $\theta' = d\theta/d\phi$.

(c) The integrand is independent of the ϕ coordinate (which is why we chose to parametrize the path as $\theta(\phi)$ and not $\phi(\theta)$) and, therefore, we can use the Beltrami identity to find the extreme of the integral.

$$F - \theta'\frac{\partial F}{\partial\theta'} = \sqrt{\theta'^2 + \sin^2\theta} - \theta'\frac{\partial}{\partial\theta'}\sqrt{\theta'^2 + \sin^2\theta} = C_1. \tag{S.54}$$

$$\Rightarrow \qquad \frac{\sin^2\theta}{\sqrt{\theta'^2 + \sin^2\theta}} = C_1 \tag{S.55}$$

This last equation is a first order differential equation, which we can bring into the following form

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \pm\frac{C_1}{\sin\theta\sqrt{\sin^2\theta - C_1^2}}. \tag{S.56}$$

The sign $\pm$ can be absorbed into the constant C_1. The solution of the last equation is

$$\phi(\theta) = \arcsin(C_1^{-1}\cot\theta) + C_2, \tag{S.57}$$

where C_2 is a constant of integration. C_1 and C_2 can be fixed by the requirement that the path passes through the points A and B.

(d) In order to see that the solution (14) describes the great circle between points A and B, we first solve it in terms of $\cot\theta$:

$$\cot\theta = C_1\sin(\phi - C_2) \tag{S.58}$$
$$\Rightarrow \qquad \cot\theta = C_1\sin\phi\cos C_1 - C_1\cos\phi\sin C_2. \tag{S.59}$$

Multiplying both sides of this equation by $R\sin\theta$ we obtain

$$R\cos\theta = \alpha R\sin\theta\sin\phi - \beta R\sin\theta\cos\phi, \tag{S.60}$$

where $\alpha = C_1\cos C_1$ and $\beta = C_1\sin C_2$. Now we can write our solution in Cartesian coordinates as

$$\beta x - \alpha y + z = 0 \tag{S.61}$$

This last expression describes the intersection of a plane in the three-dimensional space that goes through the origin of the coordinate system ($x = y = z = 0$) with the surface of the sphere at $r = R$. This intersection is a great circle on the surface of the sphere and the right choice of α and β parameters guarantees that it passes through A and B.

Note that there are two paths that connect the point A and B along the same great circle. Both of them are a solution to the differential equation (S.56) with the same boundary conditions. However, unless points A and B lie on a diameter, the length of one path is longer than the other and, therefore, does not minimize the path between the two points. This demonstrates that a solution to the Euler-Lagrange equation (or equivalent the Beltrami identity) offers only a necessary but not a sufficient conditions for the existence of an extremum.