Prologue

Given an orthonormal basis in a vector space with n dimensions, any vector can be represented by its components[1]

$$\vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i. \tag{1}$$

In order to make formulae involving vectors less cumbersome, it is very useful to adopt the *Einstein summation convention*: repeated indices are implicitly summed over and the sign that indicates the sum omitted. For instance, we shall write

$$\vec{v} = v_i \hat{e}_i, \tag{2}$$

instead of the above formula. We will also be using extensively the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

As an example, the orthonormality condition reads

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}, \tag{4}$$

and a scalar product

$$\vec{v} \cdot \vec{w} = v_i \hat{e}_i \cdot w_j \hat{e}_j = v_i w_j \delta_{ij} = v_i w_i. \tag{5}$$

Two indices that are paired and summed over as in the last step on the right are sometimes said to be *contracted*.

Exercise 1. *The Levi-Civita symbol.*

Given a vector space of dimension n, the Levi-Civita symbol is an object with n indices defined by the property

$$\varepsilon_{\dots i \dots j \dots} \equiv -\varepsilon_{\dots j \dots i \dots}\,, \tag{6}$$

together with $\varepsilon_{12\dots n} = +1$. We say that ε is *totally antisymmetric* under the exchange of any two indices.

(i) What is $\varepsilon_{i_1 \dots i_n}$ equal to when two indices take the same value?
(ii) Assuming $s_{ij} = s_{ji}$, what can you say about $\varepsilon_{\dots i \dots j \dots} s_{ij}$?
(iii) For $n = 2$, enumerate all values of the Levi-Civita symbol ε_{ij} and put them in a matrix.
(iv) For $n = 3$, list all non-zero values of the Levi-Civita symbol ε_{ijk}.

[1] In differential geometry, it is important to distinguish between upper and lower indices. For this course such distinction is not required (if you are wondering why, the reason is that we will only deal with euclidian spaces).

Solution. When two indices are equal, from the definition we get

$$\varepsilon_{\dots i \dots i \dots} = -\varepsilon_{\dots i \dots i \dots} = 0. \tag{S.1}$$

Again using the definition we get

$$\sum_{i,j} \varepsilon_{\dots i \dots j \dots} s_{ij} = \sum_{i,j} (-\varepsilon_{\dots j \dots i \dots}) s_{ij} = \sum_{i,j} (-\varepsilon_{\dots j \dots i \dots}) s_{ji} = -\sum_{k,l} \varepsilon_{\dots l \dots k \dots} s_{lk} = 0. \tag{S.2}$$

In these steps sums were written out explicitly to emphasize that, because i and j are dummy indices, their name is not really important and may be changed at one's own leisure.
The Levi-Civita symbol for a 2-dimensional vector space carries two indices, and is therefore written as ε_{ij} with $i, j \in \{1, 2\}$. From the first point we immediately get $\varepsilon_{11} = \varepsilon_{22} = 0$, by definition $\varepsilon_{12} = +1$ and from the fundamental property (6) we see that $\varepsilon_{21} = -1$. Thus, as a matrix,

$$\varepsilon = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \tag{S.3}$$

For a tridimensional vector space, the entries ε_{ijk} vanish unless all indices take different values. Therefore, by exchanging indices repeatedly, we get

$$\varepsilon_{123} = +1, \qquad \varepsilon_{213} = -1, \qquad \varepsilon_{231} = +1, \qquad \varepsilon_{321} = -1, \qquad \varepsilon_{312} = +1, \qquad \varepsilon_{132} = -1. \tag{S.4}$$

The practical examples of this course will mostly be set in euclidean space in three dimensions. Therefore we are going to work almost exclusively with ε_{ijk}, which will enable us to handle vector calculus in a very convenient way (see the next exercises). Given the following identity for the product of two Levi-Civita symbols

$$\varepsilon_{ijk}\varepsilon_{nlm} = \det \begin{pmatrix} \delta_{in} & \delta_{il} & \delta_{im} \\ \delta_{jn} & \delta_{jl} & \delta_{jm} \\ \delta_{kn} & \delta_{kl} & \delta_{km} \end{pmatrix}; \tag{7}$$

(v) Show that $\varepsilon_{ijk}\varepsilon_{ilm} = \det \begin{pmatrix} \delta_{jl} & \delta_{jm} \\ \delta_{kl} & \delta_{km} \end{pmatrix} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

(vi) Show that $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$.

(vii) Show that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

Solution. These identities all follow from one another inserting a δ and summing. We start from

$$\varepsilon_{ijk}\varepsilon_{nlm} = \delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} + \delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}; \tag{S.5}$$

using $\delta_{ab}\delta_{bc} = \delta_{ac}$ and $\delta_{aa} = 3$ repeatedly we find

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{ilm} &= \delta_{in}\varepsilon_{ijk}\varepsilon_{nlm} \\ &= 3\delta_{jl}\delta_{km} - 3\delta_{jm}\delta_{kl} + \delta_{nl}\delta_{jm}\delta_{kn} - \delta_{nl}\delta_{jn}\delta_{km} + \delta_{nm}\delta_{jn}\delta_{kl} - \delta_{nm}\delta_{jl}\delta_{kn} \\ &= 3\delta_{jl}\delta_{km} - 3\delta_{jm}\delta_{kl} + \delta_{jm}\delta_{kl} - \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl} - \delta_{jl}\delta_{km} \\ &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \end{aligned} \tag{S.6}$$

$$\varepsilon_{ijk}\varepsilon_{ijm} = \delta_{jl}\varepsilon_{ijk}\varepsilon_{ilm} = 3\delta_{km} - \delta_{km} = 2\delta_{km}, \tag{S.7}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = \delta_{km}\varepsilon_{ijk}\varepsilon_{ijm} = 3\delta_{km}. \tag{S.8}$$

One of the possible definitions of the vector product reads

$$\vec{v} \times \vec{w} \equiv \varepsilon_{ijk} v_j w_k \hat{e}_i. \tag{8}$$

(viii) Show that, also according to this definition, $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, and that its length is equal to the area spanned by a parallelogram with sides $\vec{v}$ and $\vec{w}$. Can you say to which property of ε the right-hand rule is related?

Solution. Using point (ii) we find immediately

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = \varepsilon_{ijk} v_i v_j w_k = 0, \tag{S.9}$$

which means that $\vec{v} \perp (\vec{v} \times \vec{w})$, and the same for $\vec{w}$. Applying (v) yields

$$(\vec{v} \times \vec{w})^2 = (\varepsilon_{ijk} v_j w_k)(\varepsilon_{ilm} v_l w_m) = \vec{v}^2 \vec{w}^2 - (\vec{v} \cdot \vec{w})^2 = \vec{v}^2 \vec{w}^2 (1 - \cos^2 \vartheta) = (|\vec{v}||\vec{w}| \sin \vartheta)^2, \tag{S.10}$$

which is the area of the described parallelogram. The sign of $(\vec{v} \times \vec{w})$ would always be flipped if ε had the opposite sign, as can be seen from the definition. Thus, assuming the coordinate system is right-handed, if we had taken $\varepsilon_{123} = -1$ we would ended up with the vector product being given by a *left*-hand rule.

Exercise 2. *Vector Identities*

Prove the following identities:

1. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$
2. $|\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\,|^2 = (\mathbf{a} \cdot \mathbf{c})^2\, \mathbf{b}^2 + (\mathbf{a} \cdot \mathbf{b})^2\, \mathbf{c}^2 - 2\,(\mathbf{a} \cdot \mathbf{c})\,(\mathbf{a} \cdot \mathbf{b})\,(\mathbf{b} \cdot \mathbf{c})$
3. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}$
4. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c})$
5. $\mathbf{R}\mathbf{a} \times \mathbf{R}\mathbf{b} = \mathbf{R}\,(\mathbf{a} \times \mathbf{b})$
6. $\nabla \times \nabla\psi = 0$
7. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
8. $\nabla \times (\nabla \times \mathbf{A}) = \nabla\,(\nabla \cdot \mathbf{A}) - \Delta\mathbf{A}$
9. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
10. $\nabla\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\, \mathbf{B} + (\mathbf{B} \cdot \nabla)\, \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$
11. $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\, \mathbf{A} - (\mathbf{A} \cdot \nabla)\, \mathbf{B}$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors, $\mathbf{A}$, $\mathbf{B}$ are vector fields, ψ is a function and $\mathbf{R} \in \mathrm{SO}(3)$. Moreover assume that all components A_i, B_j and also ψ are in $\mathcal{C}(2)$, i.e. two times continuously differentiable.
Don't write out cross products explicitly, but use the index notation involving the Levi-Civita symbol ε_{ijk}.

Solution.

1.

$$\begin{aligned} (\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &= \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m \\ &= \varepsilon_{kij} \varepsilon_{klm} a_j b_l c_m \\ &= a_j b_i c_j - a_j b_j c_i \\ &= (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i \end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$ and $\varepsilon_{kij} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

2. Here one can either square the previous result which is trivial or write down the cross products explicitly and contract the indices from the getgo.

3.

$$\begin{aligned}((\mathbf{a}\times\mathbf{b})\times\mathbf{c})_i &= \varepsilon_{ijk}(\varepsilon_{jlm}a_l b_m)c_k\\ &= \varepsilon_{jki}\varepsilon_{jlm}a_l b_m c_k\\ &= a_k b_i c_k - a_i b_k c_k\\ &= (\mathbf{a}\cdot\mathbf{c})b_i - (\mathbf{b}\cdot\mathbf{c})a_i\end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$ and $\varepsilon_{jki}\varepsilon_{jlm} = \delta_{kl}\delta_{im} - \delta_{km}\delta_{il}$

4.

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= \varepsilon_{ijk}a_j b_k \varepsilon_{iml}c_m d_l\\ &= a_j b_k c_j d_k - a_j b_k c_k d_j\\ &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})\end{aligned}$$

where we used $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{il}$

5. First, in the spirit of the expansion of the determinant of a matrix M we observe

$$\varepsilon_{jml}M_{ji}M_{mn}M_{ls} = \varepsilon_{ins}\det(M) \quad . \tag{S.11}$$

Hence, we find with $R^{-1} = R^T$ and $\det(R) = 1$

$$\begin{aligned}\left(R^{-1}(R\mathbf{a}\times R\mathbf{b})\right)_i &= R_{ji}(R\mathbf{a}\times R\mathbf{b})_j\\ &= R_{ji}\varepsilon_{jml}R_{mn}a_n R_{ls}b_s\\ &= \varepsilon_{ins}a_n b_s\\ &= (\mathbf{a}\times\mathbf{b})_i\end{aligned}$$

6.

$$(\nabla\times\nabla\psi)_i = \varepsilon_{ijk}\partial_j\partial_k\psi = 0$$

since the partial derivatives commute and ε_{ijk} is antisymmetric.

7.

$$\nabla\cdot(\nabla\times\mathbf{A}) = \varepsilon_{ijk}\partial_i\partial_j\mathbf{A}_k = 0$$

since the partial derivatives commute and ε_{ijk} is antisymmetric.

8.

$$\begin{aligned}(\nabla\times(\nabla\times\mathbf{A}))_i &= \varepsilon_{ijk}\varepsilon_{klm}\partial_j\partial_l\mathbf{A}_m\\ &= \varepsilon_{kij}\varepsilon_{klm}\partial_j\partial_l\mathbf{A}_m\\ &= \partial_i\partial_m\mathbf{A}_m - \partial_j\partial_j\mathbf{A}_i\\ &= (\nabla(\nabla\cdot\mathbf{A}) - \Delta\mathbf{A})_i\end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$ and $\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

9.

$$\begin{aligned}\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \partial_i(\mathbf{A}\times\mathbf{B})_i\\ &= \partial_i\varepsilon_{ijk}\mathbf{A}_j\mathbf{B}_k\\ &= \varepsilon_{ijk}\partial_i[\mathbf{A}_j\mathbf{B}_k]\\ &= \varepsilon_{ijk}(\partial_i\mathbf{A}_j)\mathbf{B}_k + \varepsilon_{ijk}\mathbf{A}_j(\partial_i\mathbf{B}_k)\\ &= \mathbf{B}_k\varepsilon_{kij}(\partial_i\mathbf{A}_j) - \mathbf{A}_j\varepsilon_{jik}(\partial_i\mathbf{B}_k)\\ &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})\end{aligned}$$

where we used $\varepsilon_{ijk} = \varepsilon_{kij}$, $\varepsilon_{ijk} = -\varepsilon_{jik}$ and the product rule for derivatives.

10.

$$
\begin{aligned}
&\left((\mathbf{A}\cdot\nabla)\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})\right)_i\\
=\;& \mathbf{A}_j\partial_j\mathbf{B}_i+\mathbf{B}_j\partial_j\mathbf{A}_i+\varepsilon_{ijk}\varepsilon_{klm}\mathbf{A}_j\partial_l\mathbf{B}_m+\varepsilon_{ijk}\varepsilon_{klm}\mathbf{B}_j\partial_l\mathbf{A}_m\\
=\;& \mathbf{A}_j\partial_j\mathbf{B}_i+\mathbf{B}_j\partial_j\mathbf{A}_i+\varepsilon_{kij}\varepsilon_{klm}\mathbf{A}_j\partial_l\mathbf{B}_m+\varepsilon_{kij}\varepsilon_{klm}\mathbf{B}_j\partial_l\mathbf{A}_m\\
=\;& \mathbf{A}_j\partial_j\mathbf{B}_i+\mathbf{B}_j\partial_j\mathbf{A}_i+\mathbf{A}_j\partial_i\mathbf{B}_j-\mathbf{A}_j\partial_j\mathbf{B}_i+\mathbf{B}_j\partial_i\mathbf{A}_j-\mathbf{B}_j\partial_j\mathbf{A}_i\\
=\;& \mathbf{A}_j\partial_i\mathbf{B}_j+\mathbf{B}_j\partial_i\mathbf{A}_j\\
=\;& \partial_i\left(\mathbf{A}_j\mathbf{B}_j\right)\\
=\;& \left(\nabla\left(\mathbf{A}\cdot\mathbf{B}\right)\right)_i
\end{aligned}
$$

where we used $\varepsilon_{ijk}=\varepsilon_{kij}$ and $\varepsilon_{kij}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$

11.

$$
\begin{aligned}
\left(\nabla\times(\mathbf{A}\times\mathbf{B})\right)_i &= \varepsilon_{ijk}\left(\partial_j(\mathbf{A}\times\mathbf{B})_k\right)\\
&= \varepsilon_{ijk}\partial_j(\varepsilon_{klm}\mathbf{A}_l\mathbf{B}_m)\\
&= \varepsilon_{ijk}\varepsilon_{klm}[(\partial_j\mathbf{A}_l)\mathbf{B}_m+\mathbf{A}_l(\partial_j\mathbf{B}_m)]\\
&= \varepsilon_{kij}\varepsilon_{klm}[(\partial_j\mathbf{A}_l)\mathbf{B}_m+\mathbf{A}_l(\partial_j\mathbf{B}_m)]\\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})((\partial_j\mathbf{A}_l)\mathbf{B}_m)+(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})(\mathbf{A}_l(\partial_j\mathbf{B}_m)\\
&= \partial_j\mathbf{A}_i\mathbf{B}_j-\partial_j\mathbf{A}_j\mathbf{B}_i+\mathbf{A}_i\partial_j\mathbf{B}_j-\mathbf{A}_j\partial_j\mathbf{B}_i\\
&= \left((\mathbf{B}\cdot\nabla)\,\mathbf{A}-\mathbf{B}\,(\nabla\cdot\mathbf{A})+\mathbf{A}\,(\nabla\cdot\mathbf{B})-(\mathbf{A}\cdot\nabla)\,\mathbf{B}\right)_i
\end{aligned}
$$

where again we used $\varepsilon_{ijk}=\varepsilon_{kij}$ and $\varepsilon_{kij}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$, as well as the product rule for differentiation.

Exercise 3. *Jacobian*

In this exercise, we will have a closer look at the Jacobian for the example of the polar coordinate transformation.

a) Calculate the Jacobian matrix $J(r,\Theta)$ for the polar coordinate transformation $x=r\cos\Theta$, $y=r\sin\Theta$.

b) Show that $dx=\cos\Theta dr-r\sin\Theta d\Theta$, $dy=\sin\Theta dr+r\cos\Theta d\Theta$ and calculate the area element $dx\times dy$ in terms of dr and $d\Theta$.

c) In the expression for $dx\times dy$, where does the Jacobian come in?

d) When does the determinant of $J(r,\Theta)$ vanish? The inverse function theorem states that if a continuously differentiable function has a non-zero Jacobian determinant at some point, the function is invertible in an open region containing that point. What does this theorem tell you about the polar coordinate transformation we looked at in this exercise?

Solution.

1. The Jacobian matrix is given by

$$
J(r,\Theta)=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\Theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\Theta}\end{pmatrix}=\begin{pmatrix}\cos(\Theta) & -r\sin(\Theta)\\ \sin(\Theta) & r\cos(\Theta)\end{pmatrix}.
$$

2. Using $d\tilde{x}_i = \frac{\partial \tilde{x}_i}{\partial x_j} dx_j$, we immediately find that $dx = \cos\Theta dr - r\sin\Theta d\Theta$ and $dy = \sin\Theta dr + r\cos\Theta d\Theta$. Now, the area calculates as

$$\begin{aligned} dx \times dy &= (\cos\Theta dr - r\sin\Theta d\Theta) \times (\sin\Theta dr + r\cos\Theta d\Theta) \\ &= r\cos^2\Theta dr \times d\Theta - r\sin^2\Theta d\Theta \times dr \\ &= r(\cos^2\Theta + \sin^2\Theta) dr \times d\Theta \\ &= r dr \times d\Theta. \end{aligned}$$

3. What we have really calculated above is $dx \times dy = (\frac{\partial x}{\partial r}\frac{\partial y}{\partial \Theta} - \frac{\partial x}{\partial \Theta}\frac{\partial y}{\partial r}) dr \times d\Theta$. Here, the expression in the brackets is none other than the determinant of the matrix J: in this case, $\det J(r, \Theta) = r$.
4. In the example of the polar coordinates, the determinant (only) vanishes at $r = 0$. The inverse function theorem tells us now that for each point except $r = 0$, an inverse transformation exists for the polar coordinate transform for an open region around that point. At $r = 0$, however, we cannot find such an inverse, as any arbitrary Θ could be assigned to that point $(x = 0, y = 0)$.

Exercise 4. *Vector Calculus*

This exercise requires using various techniques that you have learnt in this sheet.

Calculate $\partial r/\partial x_i$ where $r = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$. A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{\mathbf{a}}{r} + \frac{(\mathbf{a}\cdot\mathbf{x})\mathbf{x}}{r^3}.$$

Find the Jacobian $J_{ij} = \partial u_i/\partial x_j$ and deduce that $\nabla \cdot \mathbf{u} = 0$.

Solution. Using the suffix notation one may write $r = \sqrt{x_k x_k}$. Then

$$\frac{\partial r}{\partial x_i} = \frac{2x_k\delta_{ik}}{2\sqrt{x_k x_k}} = \frac{x_i}{r}.$$

This implies

$$\begin{aligned} J_{ij} = \frac{\partial u_i}{\partial x_j} &= -\frac{a_i x_j}{r^3} - \frac{3a_k x_k x_i x_j}{r^5} + \frac{a_k(x_k\delta_{ij} + \delta_{kj}x_i)}{r^3} \\ &= \frac{1}{r^3}(-a_i x_j + a_j x_i) - 3\frac{\mathbf{a}\cdot\mathbf{x}}{r^5}x_i x_j + \frac{\mathbf{a}\cdot\mathbf{x}}{r^3}\delta_{ij}. \end{aligned}$$

Note that this splits into a symmetric and antisymmetric part. We have $\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = J_{ii}$. If $i = j$, the antisymmetric part vanishes and so does the symmetric part once we use $x_i x_i = r^2$ and $\delta_{ii} = 3$.