Exercise 1. ***Objects rolling down a slope*** *[20 points]*

Consider two different rigid objects rolling down an inclined plane under the influence of gravity. The slope has length L and an inclination of θ with respect to the horizontal plane. The first object is a double-cone shaped body with radius R and height h. The second object is a hollow cylinder of inner radius r and outer radius R, also of height h (see sketch below). Both bodies have uniform density and the same total mass M.

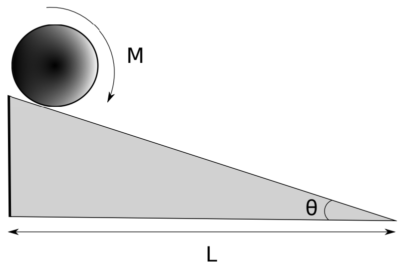

Figure 1: Illustration of the problem of rigid bodies rolling down a slope.

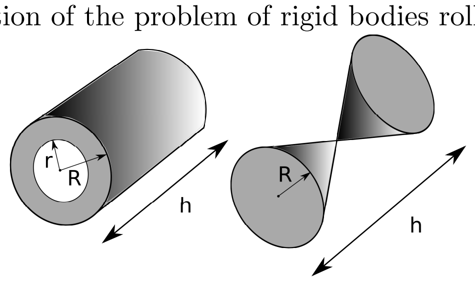

Figure 2: Illustration of the shape of the two rigid bodies considered in the first problem.

(i) Calculate the moment of inertia around the axis of rotation for the two rigid bodies. [7 points]

(ii) Introduce suitable generalised coordinates and set up the Lagrangian for the two separate bodies assuming they roll down the slope with no slipping. [4 points]

(iii) Derive the equation of motion for the system and solve it. [3 points]

(iv) Is the angular momentum conserved? Find one (other) conserved quantity of this problem. [2 points]

(v) At time $t = 0$ both objects are released at the top of the slope and let roll down. Which one of the two objects reaches the bottom of the slope first and why? Is there a way to engineer the objects in a way that they reach the bottom of the slope at the same time? [4 points]

Solution.

(i) We calculate

$$I_1 = \rho \int_0^h \int_0^{2\pi} \int_r^R \tilde{r}^2 \tilde{r} \mathrm{d}z \mathrm{d}\Phi \mathrm{d}\tilde{r} = \frac{\pi}{2}(R^4 - r^4)h\rho = \frac{M}{2}(R^2 + r^2)$$

for the hollow cylinder (its mass satisfies $M = \pi\rho h(R^2 - r^2)$) and similarly

$$I_2 = 2\int_0^{h/2}\int_0^{2\pi}\int_0^{2Rz/h} \rho\tilde{r}^3 \mathrm{d}z\mathrm{d}\Phi\mathrm{d}\tilde{r} = \pi\rho\int_0^{h/2}\frac{(2Rz)^4}{h^4}dz = \frac{\pi\rho R^4 h}{10} = \frac{3MR^2}{10}$$

for the double-cone-shaped object whose mass satisfies

$$M = 4\pi\rho\int_0^{h/2}\int_0^{2Rz/h}\tilde{r}\mathrm{d}z\mathrm{d}\tilde{r} = \frac{2}{3}\pi\rho\left(\frac{2R}{h}\right)^2\left(\frac{h}{2}\right)^3 = \frac{\pi\rho h R^2}{3}.$$

(ii) To determine the Lagrangian of the problem, we introduce two generalised coordinates describing the instantaneous motion of the system. The first coordinate x is the distance of the body from the top of the slope, while the second coordinate is the angle ϕ of rotation of the body about its symmetry axis (see sketch). Since the objects roll without slipping, the two coordinates are related by the condition

$$R\phi = x, \tag{S.1}$$

which amounts to say that the full circumference of the objects "roll out" onto the slope. The kinetic energy of the bodies can be separated in two components T_T and T_R, corresponding to the (translational) kinetic energy of the center of mass and the rotational energy of the body around the center of mass. The translational component depends on the linear velocity of the object, while the rotational component depends on the angular velocity of the rotation:

$$T = T_T + T_R \tag{S.2}$$
$$= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \tag{S.3}$$
$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2. \tag{S.4}$$

The moments of inertia I of the solids have been computed in section (a). The potential energy is computed from gravity as

$$V = Mgy = Mg(L\tan\theta - x\sin\theta + R\cos\theta), \tag{S.5}$$

but the constant terms do not play a role in the derivation of the equations of motion, so we might as well write simply

$$V = -Mgx\sin\theta. \tag{S.6}$$

The Lagrangian of the system is therefore written as

$$\mathcal{L}(x, \dot{x}, \phi, \dot{\phi}) = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + Mgx\sin\theta. \tag{S.7}$$

With the rolling condition (S.1) the dependence of the Lagrangian can be reduced to a generalised coordinate only, which we choose to be x:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}\left(M + \frac{I}{R^2}\right)\dot{x}^2 + Mgx\sin\theta. \tag{S.8}$$

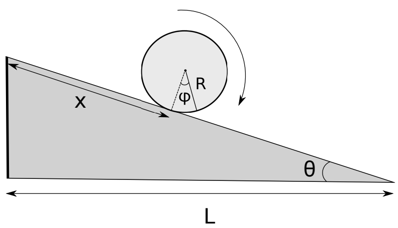

Figure 3: Sketch of the coordinates used in the problem of a rigid body rolling down a slope.

(iii) The equation of motion is computed from the Euler-Lagrange equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} \tag{S.9}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M + \frac{I}{R^2}\right)\dot{x} = Mg\sin\theta \tag{S.10}$$
$$\ddot{x} = \frac{Mg\sin\theta}{M + \frac{I}{R^2}} \tag{S.11}$$

The solution of this equation is obtained directly by twofold integration:

$$x(t) = At^2 + V_0 t + \frac{H - H_0}{\sin\theta} \tag{S.12}$$

with V_0 and H_0 being integration constants corresponding to the initial velocity and to the initial height of the object, $H = L\tan\theta$ being the total height of the slope, and $A = \frac{g\sin\theta}{1+\frac{I}{MR^2}}$ corresponding to the acceleration.

(iv) The only conserved quantity is the energy $E = T + V$, because the variable ϕ is *not* ignorable as it is related to x via the rolling condition. Therefore the conjugated momentum to ϕ (the angular momentum) is *not* conserved.

(v) The total time T of the path is obtained simply by solving for T in the equation

$$x_{tot}(T) = AT^2 \tag{S.13}$$

$$\frac{L}{\cos\theta} = AT^2, \tag{S.14}$$

which follows from the initial conditions (no initial velocity $V_0 = 0$ and initial height $H_0 = H$). The time T is therefore

$$T = \sqrt{\frac{L}{A\cos\theta}} \tag{S.15}$$

$$= \sqrt{\frac{L(1+\frac{I}{MR^2})}{g\sin\theta\cos\theta}}. \tag{S.16}$$

We see that the body with the *smallest* moment of inertia will arrive first at the bottom of the slope. However, it is not possible to predict in advance which solid has the smallest moment of inertia, unless the geometrical parameters of the bodies are known. The moments of inertia with respect to the symmetry axis were computed in (a) and are:

$$I_{DC} = \frac{3}{10}MR^2 \tag{S.17}$$

$$I_C = \frac{1}{2}M(r^2 + R^2) \tag{S.18}$$

We see that for any value of r, the moment of inertia of the double-cone is smaller than the moment of inertia of the hollow cylinder and so the double-cone arrives first. Therefore, there is no way to engineer the solids in a way that they arrive at the end of the slope at the same time, except for changing their masses.

Exercise 2. ***Particle on cylinder*** *[20 points]*

Consider a cylinder of radius R laying horizontally (and fixed) on the ground. A point-like particle of mass m is initially at rest on top of it and then starts falling. The only external force acting on the cylinder is gravity. Take the origin to be the centre of the cylinder.

The purpose of this exercise is to find the angle at which the particle falls off the cylinder.

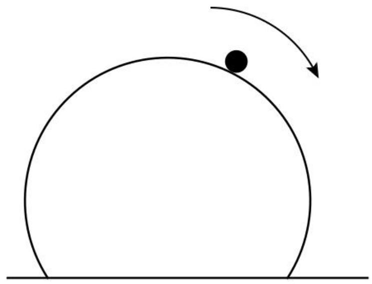

Figure 4: The particle on the cylinder.

(i) First derive the Euler-Lagrange equations describing the system.
Hint: the points which follow will be easier if you introduce a Lagrange multiplier λ to describe the constraint imposed on the motion of the particle.

[10 points]

Solution. In polar coordinate, with the centre of the cylinder as origin and with θ being the angle the position vector forms the the vertical, the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\cos\theta \tag{S.19}$$

The particle can't pass through the cylinder, so that, as long it doesn't detach from it, we have the constraint $r = R$, i.e. $f(r) = r - R = 0$, where r is the distance of the particle from the center of the hoop. The Euler-Lagrange Equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \lambda\frac{\partial f}{\partial q_i}, \tag{S.20}$$

with the generalized coordinates q_i being, in our case, r and θ. By taking the derivatives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r} \tag{S.21}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta \tag{S.22}$$

$$\lambda\frac{\partial f}{\partial r} = \lambda \tag{S.23}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2 + 2mr\dot{r}\dot{\theta} \tag{S.24}$$

$$\frac{\partial L}{\partial \theta} = mgr\sin\theta \tag{S.25}$$

$$\lambda\frac{\partial f}{\partial \theta} = 0, \tag{S.26}$$

we obtain, respectively, the equations of motion

$$m\left(\ddot{r} - r\dot{\theta}^2 + g\cos\theta\right) = \lambda \tag{S.27}$$

and

$$-\,mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} + mgr\sin\theta = 0\,. \tag{S.28}$$

(ii) Now find the angle at which the particle drops off the cylinder.

a) Rewrite one of the equations of motion that you found in the previous part by setting $r = R$ to obtain a differential equation in θ which does not involve λ. Use the chain rule to show that

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\dot{\theta} \tag{1}$$

and use this relation, together with the equations of motion, to obtain an expression for $\dot{\theta}$. Note that there is an integration constant to be determined: use the proper initial conditions for this.

Hint: you should get

$$\dot{\theta}^2 = -2\frac{g}{R}(\cos\theta - 1)\,. \tag{2}$$

[7 points]

Solution. To do this we should write λ as a function of *only* θ, by combining ($S.27$) and ($S.28$): of course $r = R$, so it is constant, therefore Eqs. ($S.27$) and ($S.28$) simplify considerably:

$$m\,(-R\dot{\theta}^2 + g\,\cos\theta) \;=\; \lambda \tag{S.29}$$

$$\ddot{\theta} \;=\; \frac{g}{R}\sin\theta \tag{S.30}$$

By rewriting the second derivative in the r.h.s. of Eq. (S.30)

$$\ddot{\theta} \;=\; \frac{d\dot{\theta}}{dt} \;=\; \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{d\theta}\dot{\theta} = \frac{1}{2}\frac{d}{d\theta}\dot{\theta}^2 \tag{S.31}$$

$$\Longrightarrow\; \dot{\theta}^2 \;=\; 2\int d\theta\,\ddot{\theta} = 2\frac{g}{R}\int d\theta\,\sin\theta \;=\; -2\frac{g}{R}\cos\theta \,+\, const \tag{S.32}$$

The constant is fixed by requiring $\theta = 0$ and $\dot{\theta} = 0$ yielding

$$\dot{\theta}^2 \;=\; -2\frac{g}{R}(\cos\theta \,-\, 1) \tag{S.33}$$

b) Using the above expression for $\dot{\theta}$ and noting that the constraint λ becomes zero when the particle is not influenced by the cylinder, find the angle at which the particle falls off the cylinder.

[3 points]

Solution. Plugging Eq. (S.33) into Eq. (S.29):

$$m\,(2R\frac{g}{R}(\cos\theta - 1) + g\,\cos\theta) \;=\; \lambda, \tag{S.34}$$

$$\Rightarrow \quad mg\,(3\cos\theta \,-\, 2) = \lambda \tag{S.35}$$

The angle at which λ goes to zero is where the particle doesn't feel the reaction force from the cylinder, and therefore is free to detach from it. This when

$$\cos\theta = \frac{2}{3} \quad\Rightarrow\quad \theta = 48.2\ldots^o \tag{S.36}$$

Exercise 3. ***Coupled masses on a circular guide*** *[20 points]*

(i) Given the action functional

$$S = \int L(q_i, \dot{q}_i, t)\, dt, \tag{3}$$

present a short derivation of the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0, \tag{4}$$

using the least action principle, i.e. by extremizing S. [4 points]

Solution. In correspondence to a variation of the motion

$$q_i(t) \to q_i(t) + \delta q_i(t), \tag{S.37}$$

the action changes by

$$\delta S = \int \delta L(q_i, \dot{q}_i, t)\, \mathrm{d}t = \int \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] \mathrm{d}t. \tag{S.38}$$

The variation of velocities is related to δq_i by

$$\delta \dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\delta q_i, \tag{S.39}$$

therefore integrating by part and using the boundary conditions of the problem to argue that δq_i must vanish at the integration extrema yields

$$\delta S = \int \left[\left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right] \mathrm{d}t. \tag{S.40}$$

The fundamental lemma of the calculus of variations then implies that, if δS is to be zero for every possible δq_i,

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{S.41}$$

Consider two identical masses m moving on a circular guide with radius R, subject to gravity. The two masses are coupled by a spring of stiffness $k = mg/R$, negligible mass and rest length $l_0 = \theta_0 R$, which is also wrapped around the guide.

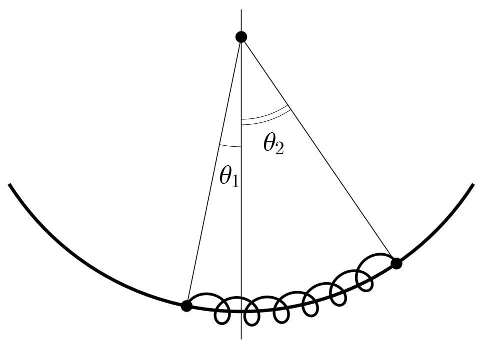

Figure 5: system of two coupled masses on a circular guide.

(ii) Using the angles θ_1, θ_2 from the downward vertical as coordinates, show that, for small θ_1, θ_2 and θ_0, the Lagrangian is given by:

$$L(\theta_1, \theta_2; \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}mR^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{1}{2}mgR(\theta_1^2 + \theta_2^2) - \frac{1}{2}mgR(\theta_1 - \theta_2 - \theta_0)^2, \tag{5}$$

up to an irrelevant additive constant. [4 points]

Solution. The kinetic term simply reads

$$T = \frac{1}{2} m R^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2). \tag{S.42}$$

The gravitational potential energy is

$$U_g = mgR(1 - \cos\theta_1) + mgR(1 - \cos\theta_2), \tag{S.43}$$

up to an irrelevant additive constant. Moreover, the potential energy of the spring can be easily written as

$$U_s = \frac{1}{2} k \Delta l^2 = \frac{1}{2} k [R(\theta_1 - \theta_2) - l_0]^2 = \frac{1}{2} mgR(\theta_1 - \theta_2 - \theta_0)^2. \tag{S.44}$$

The complete lagrangian therefore reads

$$L = T - U_g - U_s = \frac{1}{2} m R^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) - mgR(2 - \cos\theta_1 - \cos\theta_2) - \frac{1}{2} mgR(\theta_1 - \theta_2 - \theta_0)^2. \tag{S.45}$$

If all angles are small, the cosines can be expanded according to

$$\cos\theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4), \tag{S.46}$$

to find equation (5).

(iii) Let now $\omega^2 = g/R$. Find the equations of motion for this system in terms of θ_1, θ_2 and their derivatives. [2 points]

Solution. The Euler-Lagrange equations read

$$\begin{cases} \dfrac{\partial L}{\partial \theta_1} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial L}{\partial \dot{\theta}_1} = -mgR\theta_1 - mgR(\theta_1 - \theta_2 - \theta_0) - mR^2 \ddot{\theta}_1 = 0, \\ \dfrac{\partial L}{\partial \theta_2} - \dfrac{\mathrm{d}}{\mathrm{d}t} \dfrac{\partial L}{\partial \dot{\theta}_2} = -mgR\theta_2 + mgR(\theta_1 - \theta_2 - \theta_0) - mR^2 \ddot{\theta}_2 = 0. \end{cases} \tag{S.47}$$

Introducing the definition of ω one thus gets

$$\begin{cases} \ddot{\theta}_1 = -\omega^2 (2\theta_1 - \theta_2 - \theta_0), \\ \ddot{\theta}_2 = -\omega^2 (\theta_1 - 2\theta_2 - \theta_0). \end{cases} \tag{S.48}$$

(iv) Determine the equilibrium values for θ_1, θ_2 by requiring the equations of motion to hold when the angles are constant. Get rid of the constant terms in the equations of motion by appropriately shifting θ_1 and θ_2 to some new coordinates ϑ_1 and ϑ_2. You should find

$$\begin{cases} \ddot{\vartheta}_1 = \omega^2 (-2\vartheta_1 + \vartheta_2), \\ \ddot{\vartheta}_2 = \omega^2 (-2\vartheta_2 + \vartheta_1). \end{cases} \tag{6}$$

[4 points]

Solution. If the system admits an equilibrium position, i.e. a constant solution $\theta_1 = \theta_1^{\mathrm{eq}}$, $\theta_2 = \theta_2^{\mathrm{eq}}$, this corresponds to vanishing time derivatives (and second derivatives). The equations of motion in this case read

$$\begin{cases} 0 = -\omega^2 (2\theta_1^{\mathrm{eq}} - \theta_2^{\mathrm{eq}} - \theta_0), \\ 0 = -\omega^2 (\theta_1^{\mathrm{eq}} - 2\theta_2^{\mathrm{eq}} - \theta_0), \end{cases} \tag{S.49}$$

which can be promptly solved

$$\begin{cases} 2\theta_1^{\mathrm{eq}} - \theta_2^{\mathrm{eq}} = \theta_0, \\ \theta_1^{\mathrm{eq}} - 2\theta_2^{\mathrm{eq}} = \theta_0, \end{cases} \Rightarrow \begin{cases} 3\theta_1^{\mathrm{eq}} - 3\theta_2^{\mathrm{eq}} = 2\theta_0, \\ \theta_1^{\mathrm{eq}} + \theta_2^{\mathrm{eq}} = 0, \end{cases} \Rightarrow \begin{cases} \theta_1^{\mathrm{eq}} = \theta_0/3, \\ \theta_2^{\mathrm{eq}} = -\theta_0/3. \end{cases} \tag{S.50}$$

Defining $\vartheta_i = \theta_i - \theta_i^{\mathrm{eq}}$ in the equations of motion then gives back (6).

For small deviations from the equilibrium position, independent oscillatory motions that happen each with its single characteristic frequency (called *normal modes*) can be determined by diagonalizing the equations of motion or, if the system is simple enough, inferred from physical considerations.

(v) Describe, with words and sketches, the normal modes of oscillation you expect to see in this simple system.

[2 points]

Solution. If the system of two masses and a spring collectively oscillates, the spring itself plays no role whatsoever in the motion since its length is constant. This therefore defines a normal mode where the center of mass oscillates back and forth without any change in the configuration of the two masses around it. Conversely, the center of mass could stay still and the two masses symmetrically oscillate around it, which defines a motion where the spring is maximally involved. Given that the influence of gravity is the same on both modes and the second has an additional coupling, a higher frequency is expected for the

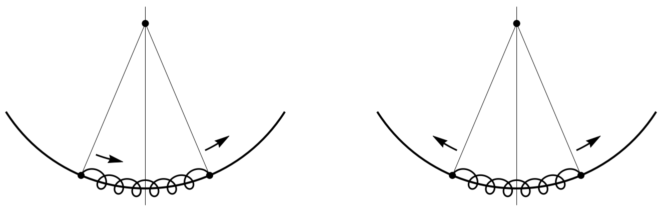

Figure 6: Sketches of the two expected normal modes.

(vi) Decouple the equations of motion (6) and show that the natural frequencies of the system are ω and $\sqrt{3}\omega$.

[4 points]

Solution. The quickest way to decouple the system (6) is just to take the sum and difference of the two equations:

$$\begin{cases} \ddot{\vartheta}_1 + \ddot{\vartheta}_2 = -\omega^2(\vartheta_1 + \vartheta_2), \\ \ddot{\vartheta}_1 - \ddot{\vartheta}_2 = -3\omega^2(\vartheta_1 - \vartheta_2); \end{cases} \tag{S.51}$$

or, defining $\vartheta_\pm = \vartheta_1 \pm \vartheta_2$,

$$\begin{cases} \ddot{\vartheta}_+ = -\omega^2\vartheta_+, \\ \ddot{\vartheta}_- = -3\omega^2\vartheta_-. \end{cases} \tag{S.52}$$

These are two decoupled harmonic oscillator equations with frequencies ω (for the motion of the center of mass, in phase) and $\sqrt{3}\omega$ (for compression and decompression of the spring, anti-phase).

Exercise 4. *A constant time shift as a canonical transformation* *[20 points]*

Consider a harmonic oscillator described by the following Hamiltonian:

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}, \tag{7}$$

where m is the mass and ω is the frequency of the oscillator.

(i) Solve Hamilton equations of motion and show that

$$q(t) = q_0 \sin(\omega t + \phi) \qquad \text{and} \qquad p(t) = m\omega q_0 \cos(\omega t + \phi). \tag{8}$$

Here q_0 and ϕ are integration constants. [4 points]

Now introduce transformation of variables as $Q(t) = q(t + \tau)$ and $P(t) = p(t + \tau)$, where τ is a constant shift in time.

(ii) Show that the above transformation is equivalent to the following one

$$Q = \cos(\omega\tau)q + \frac{\sin(\omega\tau)}{m\omega}p, \tag{9}$$

$$P = -m\omega \sin(\omega\tau)q + \cos(\omega\tau)p. \tag{10}$$

[4 points]

(iii) Using Poisson brackets or otherwise, check that the last transformation in (ii) is a canonical transformation. [2 points]

(iv) By inverting (9)-(10), express q and p in terms of Q and P and show that the new Hamiltonian $K(Q, P)$ has exactly the same form as $H(q, p)$:

$$K(Q, P) = \frac{P^2}{2m} + \frac{m\omega^2 Q^2}{2}. \tag{11}$$

[4 points]

(v) Find a generating function $F(q, Q)$ of the canonical transformation (9)-(10). [6 points]

Hint. Use the relations $p = \frac{\partial F}{\partial q}$ and $P = -\frac{\partial F}{\partial Q}$. Also from (9)-(10) express p and P in terms of q and Q. Integrate p as a function of q to obtain q-dependence of F . Remember to include the undetermined integrating constant $C(Q)$. Use the equation for $P(q, Q)$ to find $C(Q)$ and determine the final form of $F(q, Q)$.

Solution.

(i) The Hamilton's equations are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \ \dot{p} = -\frac{\partial H}{\partial q} = m\omega^2 q \tag{S.53}$$

Differentiating the equation for q gives

$$\ddot{q} = \frac{\dot{p}}{m} = -\omega^2 q. \tag{S.54}$$

This is an equation for harmonic oscillator, which solution can be written in the form $q(t) = q_0 \sin(\omega t + \phi)$ for some constants q_0 and ϕ. Now the first Hamilton's equation implies $p = m\dot{q} = m\omega q_0 \cos(\omega t + \phi)$.

(ii) Doing little algebra

$$\begin{aligned} Q(t) &= q(t+\tau) = q_0 \sin(\omega t + \phi + \omega\tau) = q_0 \sin(\omega t + \phi)\cos(\omega\tau) + q_0 \cos(\omega t + \phi)\sin(\omega\tau) = \\ &= \cos(\omega\tau)q + \frac{\sin(\omega\tau)}{\omega m}p, \\ P(t) &= p(t+\tau) = m\omega q_0 \cos(\omega t + \phi + \omega\tau) = m\omega q_0 \cos(\omega t + \phi_0)\cos(\omega\tau) - m\omega q_0 \sin(\omega t + \phi)\sin(\omega\tau) = \\ &= -m\omega \sin(\omega\tau)q + \cos(\omega\tau)p \end{aligned} \tag{S.55}$$

(iii) Computing the Poisson brackets gives

$$[Q,P]_{q,p} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \cos(\omega\tau)\cos(\omega\tau) - \frac{\sin(\omega\tau)}{\omega m}[-m\omega\sin(\omega\tau)] = \cos^2(\omega\tau) + \sin^2(\omega\tau) = 1, \tag{S.56}$$

so the transformation is canonical.

(iv) The canonical transformation can be written in matrix form

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \cos(\omega\tau) & \frac{1}{\omega m}\sin(\omega\tau) \\ -\omega m\sin(\omega\tau) & \cos(\omega\tau) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \tag{S.57}$$

which is easy to invert because the matrix has determinant 1 and so

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos(\omega\tau) & -\frac{1}{\omega m}\sin(\omega\tau) \\ \omega m\sin(\omega\tau) & \cos(\omega\tau) \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} \tag{S.58}$$

The Hamiltonian becomes in terms of the new coordinates

$$K(Q,P) = \frac{1}{2m}\left[\omega m\sin(\omega\tau)Q + \cos(\omega\tau)P\right]^2 + \frac{1}{2}m\omega^2\left[\cos(\omega\tau)Q - \frac{1}{\omega m}\sin(\omega\tau)P\right]^2 = \tag{S.59}$$

$$= \frac{1}{2}m\omega^2Q^2\left[\sin^2(\omega\tau)+\cos^2(\omega\tau)\right] + \frac{P^2}{2m}\left[\cos^2(\omega\tau)+\sin^2(\omega\tau)\right] + \omega PQ\sin(\omega\tau)\cos(\omega\tau) - \tag{S.60}$$

$$- \omega PQ\sin(\omega\tau)\cos(\omega\tau) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2Q^2 \tag{S.61}$$

(v) First we express

$$p(q,Q) = \frac{m\omega}{\sin(\omega\tau)}\left[Q - \cos(\omega\tau)q\right] \tag{S.62}$$

From the relation $p = \partial F/\partial q$ one can extract

$$F = \int p(q,Q)\,\mathrm{d}q = \frac{m\omega}{\sin(\omega\tau)}\left[qQ - \frac{1}{2}\cos(\omega\tau)q^2\right] + C(Q), \tag{S.63}$$

where the function $C(Q)$ acts as a 'constant' of integration with respect to q. One can also find

$$P(q,Q) = -\omega m\sin(\omega\tau)q + \frac{m\omega}{\tan(\omega\tau)}\left[Q - \cos(\omega\tau)q\right] = -\frac{m\omega}{\sin(\omega\tau)}q + \frac{m\omega}{\tan(\omega\tau)}Q \tag{S.64}$$

Now $P = -\partial F/\partial Q$ implies

$$C'(Q) = -\frac{m\omega}{\tan(\omega\tau)}Q \Rightarrow C(Q) = -\frac{m\omega}{2\tan(\omega\tau)}Q + \text{const.} \tag{S.65}$$

Putting everything together gives the generating function

$$F(q,Q) = -\frac{m\omega}{2\tan(\omega\tau)}(q^2+Q^2) + \frac{m\omega}{\sin(\omega\tau)}qQ + \text{const.} \tag{S.66}$$