## Exercise 1. *Routh's Procedure*

In this exercise you will learn to apply the Routh's procedure to a central potential problem.

Suppose we have a system of $n$ degrees of freedom, and $n-s$ coordinates are cyclic. We label them as $q_{s+1}, \cdots, q_n$. Then we introduce the Routhian:

$$R\left(q_1, \cdots, q_n; \dot{q}_1, \cdots, \dot{q}_s; p_{s+1}, \cdots, p_n; t\right) = \sum_{i=s+1}^{n} p_i \dot{q}_i - \mathcal{L} \tag{1}$$

$$= H_{\text{cycl}}\left(p_{s+1}, \cdots, p_n\right) - \mathcal{L}_{\text{noncycl}}\left(q_1, \cdots, q_s; \dot{q}_1, \cdots, \dot{q}_s\right) \tag{2}$$

Then it is apparent that

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i} = 0 \qquad i = 1, \cdots, s \tag{3}$$

$$\frac{\partial R}{\partial p_i} = \dot{q}_i, \quad \frac{\partial R}{\partial q_i} = -\dot{p}_i = 0, \qquad i = s+1, \cdots, n \tag{4}$$

so the Routhian is Hamiltonian on the cyclic variables and Lagrangian on the non-cyclic ones.

In order to understand the Routh's procedure better, consider the Lagrangian for central potential:

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{5}$$

where

$$U(r) = -\frac{k}{r^n} \tag{6}$$

a) Determine the cyclic variable and write down the Routhian.

b) Apply the Euler-Lagrange equations (3) to the noncyclic coordinate to obtain the equation of motion.

c) Apply now the Hamilton's equation (4) to the cyclic variable.

## Exercise 2. *Canonical Transformations*

The transformation equations between two sets of coordinates are

$$Q = \log(1 + q^{1/2}\cos p) \tag{7}$$

$$P = 2(1 + q^{1/2}\cos p)q^{1/2}\sin p. \tag{8}$$

a) Using the symplectic criterion, show from these transformation equations that $Q, P$ are canonical variables if $q$ and $p$ are.

b) Show the same thing using Poisson-brackets.

c) Show that the function that generates this transformation is

$$F_3 = -(e^Q - 1)^2 \tan p.$$

**Exercise 3.** ***Canonical transformations with two coordinates***

Consider the transformation

$$\begin{cases} Q_1 = q_1, \\ Q_2 = p_2, \end{cases} \qquad \begin{cases} P_1 = p_1 - 2p_2, \\ P_2 = -2q_1 - q_2. \end{cases} \tag{9}$$

Prove that such a transformation is canonical

a) using the symplectic criterion,
b) using Poisson brackets.

**Exercise 4.** ***Hamiltonian with dissipative force***

A particle of mass $m$ moves in one dimension $q$ in a potential $V(q)$ and is subject to a damping force $F = -2m\gamma\dot{q}$ proportional to its velocity.

a) Show that the equation of motion can be obtained from the lagrangian

$$L[q, \dot{q}, t] = e^{2\gamma t}\left[\frac{1}{2}m\dot{q}^2 - V(q)\right]. \tag{10}$$

b) Compute the canonical momentum $p$ conjugate to $q$ and find the Hamiltonian $H[q, p, t]$.
c) Using the generating function

$$F_2(q, P, t) = qPe^{\gamma t}, \tag{11}$$

find the transformed Hamiltonian $K[Q, P, t]$.

Now consider a harmonic oscillator of potential

$$V(q) = \frac{1}{2}m\omega^2 q^2. \tag{12}$$

d) Which of the Hamiltonians $H$ and $K$ is a constant of motion? Why?
e) In the underdamped case $\gamma < \omega$, obtain the solution $q(t)$ and express the integration constant related to the oscillation amplitude in terms of the conserved quantity.